

## ON A CERTAIN METHOD OF SUCCESSIVE APPROXIMATIONS

BY

M. KWAPISZ (GDAŃSK)

The purpose of this paper is to discuss two methods of successive approximations which were used by Picone [5]-[7] to prove the existence and uniqueness of solutions of integral equations. These methods were considered in general form by Ważewski [10]. Sokolov [8], [9], Lucka [4], Kurpiel [2], Kwapisz [3] have also dealt with these methods.

The first method is the Picard's method of successive approximations — in abbreviation PM — and the second one is a modification of PM — in abbreviation MPM. In this paper we shall consider operator equations in a Banach space which are generalizations of the equations considered by Picone.

It will be shown that under assumption analogous to those of Picone both sequences constructed by PM and MPM are convergent to the unique solution of the operator equation we shall deal with.

It will also be shown that under some additional assumptions MPM yields a better error estimation than PM.

1. Let us consider a Banach space  $B$  with the norm  $\|\cdot\|$ , some closed set  $D \subset B$ ,  $f \in B$ , an operator (not necessarily linear)  $A$ ,  $u \in D$ ,  $u \rightarrow Au \in B$ , and the operator equation

$$(1) \quad u = Au + f.$$

Let us quote the well-known Banach theorem:

THEOREM 1. *If for any two  $u, u' \in D$  the Lipschitz condition*

$$(2) \quad \|Au - Au'\| \leq q \|u - u'\|$$

*is fulfilled for some  $0 < q < 1$  and if for some  $u_0 \in D$  all iterations*

$$(3) \quad u_{n+1} = Au_n + f, \quad n = 0, 1, \dots,$$

*belong to  $D$ , then the sequence  $\{u_n\}$  is convergent to a solution  $u$  of (1) which is unique in  $D$  and for which the estimations*

$$\|u - u_n\| \leq \frac{q^n}{1-q} \cdot U, \quad n = 0, 1, \dots,$$

hold true, where

$$U = \|f + Au_0 - u_0\|.$$

A sequence  $\{u_n\}$  constructed according to (3) for a given  $u_0 \in D$  and an operator  $A$  is said to be *constructed* by PM.

Remark 1. If  $A$  is a linear operator, then we suppose that it is defined in the whole space  $B$  and that its norm  $\|A\|$  is less than 1. In this case equation (1) has a unique solution  $u$  in the whole space  $B$  and the estimations

$$(4) \quad \|u\| \leq \frac{\|f\|}{1 - \|A\|},$$

$$(5) \quad \|u - f\| \leq \frac{\|A\|}{1 - \|A\|} \cdot \|f\|$$

hold true.

Remark 2. If we suppose that  $D$  contains with  $u_0$  also the sphere

$$K: \|u - u_0\| \leq \frac{U}{1 - q},$$

then  $u_n \in D$  for all  $n = 0, 1, \dots$ , because in this case we have

$$\|u_n - u_0\| \leq \frac{U}{1 - q}, \quad n = 0, 1, \dots$$

2. We shall now consider a family of linear operators

$$(6) \quad H(v)u,$$

where the parameter  $v$  runs over  $D$ .

Let us suppose that for any two  $v, v' \in D$  we have

$$(7) \quad h = \sup_{v \in D} \|H(v)\| < 1,$$

and

$$(8) \quad \|H(v) - H(v')\| \leq L\|v - v'\|.$$

LEMMA A. If assumptions (6)-(8) are satisfied and  $u, u'$  are solutions of the equations

$$(9) \quad u = f + H(v)u, \quad u' = f' + H(v')u',$$

then the estimations

$$(10) \quad \|u\| \leq \frac{\|f\|}{1 - h}, \quad \|u'\| \leq \frac{\|f'\|}{1 - h},$$

$$(11) \quad \|u - u'\| \leq \frac{\|f - f'\|}{1 - h} + \frac{\min(\|f\|, \|f'\|)}{(1 - h)^2} L\|v - v'\|$$

hold true.

Proof. (10) follows immediately from Remark 1 (cf. estimation (4)). Subtracting the second equation of (9) from the first one we obtain

$$\begin{aligned}\|u - u'\| &= \|f + H(v)u - f' - H(v')u'\| \\ &\leq \|f - f'\| + \|H(v)u - H(v)u' + H(v)u' - H(v')u'\| \\ &\leq \|f - f'\| + \|H(v)\| \|u - u'\| + \|u'\| \cdot L \cdot \|v - v'\|.\end{aligned}$$

Hence

$$\|u - u'\| \leq \frac{\|f - f'\|}{1 - h} + \frac{L \|f'\|}{(1 - h)^2} \|v - v'\|.$$

Similarly we infer that

$$\|u - u'\| \leq \frac{\|f - f'\|}{1 - h} + \frac{L \|f\|}{(1 - h)^2} \|v - v'\|.$$

The two last inequalities imply (11).

THEOREM 2. Suppose assumptions (7) and (8) are satisfied. If

$$(12) \quad \alpha = \frac{L \|f\|}{(1 - h)^2} < 1$$

and if for some  $u_0 \in D$

$$(13) \quad u_{n+1} = f + H(u_n)u_{n+1}, \quad n = 0, 1, \dots,$$

belong to  $D$ , then

$$u = \lim_{n \rightarrow \infty} u_n$$

exists and it is the unique solution of

$$(14) \quad u = f + H(u)u$$

in  $D$ . Moreover, we have

$$(15) \quad \|u - f\| \leq \frac{h}{1 - h} \|f\|,$$

$$(16) \quad \|u\| \leq \frac{\|f\|}{1 - h},$$

$$(17) \quad \|u - u_n\| \leq \alpha^n \|u - u_0\| \leq \alpha^n \left[ \|f - u_0\| + \frac{h \|f\|}{1 - h} \right].$$

Proof. (15) and (16) follow immediately from (4) and (5), respectively.

By virtue of (11) we have

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \alpha \|u_n - u_{n-1}\|, \\ \|u_{n+1} - u_n\| &\leq \alpha^n \|u_1 - u_0\|.\end{aligned}$$

Hence, by (12), the sequence  $\{u_n\}$  is convergent to a solution  $u$  of (14). The uniqueness of  $u$  and estimation (17) are also a direct consequence of Lemma A.

A sequence  $\{u_n\}$  constructed according to (13) for a given  $u_0 \in D$  and a family  $H(v)$  of operators is said to be *constructed by MPM*.

Remark 3. If we suppose that together with  $u_0$  the set  $D$  contains also the sphere

$$K': \|u - u_0\| \leq \|f - u_0\| + \frac{h \|f\|}{1 - h},$$

then  $u_n \in D$  for all  $n = 0, 1, \dots$ , because in this case

$$\|u_n - u_0\| \leq \|f - u_0\| + \frac{h \|f\|}{1 - h}, \quad n = 0, 1, \dots$$

THEOREM 3. *If the assumptions of Theorem 2 are satisfied and  $D$  contains the sphere*

$$S: \|u\| \leq r = \frac{\|f\|}{1 - h},$$

*then the operator*

$$A_1 u = H(u)u,$$

*defined in  $D$ , satisfies in  $S$  the Lipschitz condition with constant  $q = h + rL < 1$ .*

Proof. If  $u, u' \in S$ , then we have

$$\begin{aligned} \|A_1 u - A_1 u'\| &= \|H(u)u - H(u')u\| \\ &= \|H(u)u - H(u')u + H(u')u - H(u')u'\| \\ &\leq \|u\| \|H(u) - H(u')\| + \|H(u')\| \|u - u'\| \\ &\leq (h + rL) \|u - u'\| = q \|u - u'\|. \end{aligned}$$

By (12) we have  $q < 1$ .

3. Put

$$(18) \quad \tilde{u}_0 = 0, \quad \tilde{u}_{n+1} = H(\tilde{u}_n)\tilde{u} + f, \quad n = 0, 1, \dots$$

THEOREM 4. *If the assumptions of Theorem 3 are satisfied, then both sequences  $\{u_n\}$  and  $\{\tilde{u}_n\}$  defined by (13) and (18), respectively, with  $u_0 = \tilde{u}_0 = \Theta$  converge and*

$$u = \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \tilde{u}_n$$

*is the unique solution of (14) in  $D$ .*

Moreover, we have

$$(19) \quad \|u - u_n\| \leq a^n \frac{\|f\|}{1-h},$$

$$(20) \quad \|u - \tilde{u}_n\| \leq q^n \frac{\|f\|}{1-h},$$

$$(21) \quad 0 \leq a < q < 1.$$

**Proof.** Inequalities (19) and (20) follow immediately from Theorems 1, 2 and 3. It remains to prove that  $a < q$ . Let  $\beta = L\|f\|$ . Since  $a < 1$ , we have successively

$$\begin{aligned} \beta &< (1-h)^2, & \beta h &< (1-h)^2 h, \\ \beta &< h(1-h)^2 + \beta(1-h), \\ a &= \frac{\beta}{(1-h)^2} < h + \frac{\beta}{1-h} = q. \end{aligned}$$

Observe that inequality (21) does not imply that there is no error estimation better than (20).

In fact, if an operator  $C(u, v), (u, v) \rightarrow C(u, v) \in S$ , where  $u, v \in S$ , satisfies the Lipschitz condition

$$\|C(u, v) - C(u', v')\| \leq \mu[\|u - u'\| + \|v - v'\|],$$

then the operator

$$C_1(u) = C(u, u)$$

satisfies in  $S$  the Lipschitz condition with constant  $\nu$  such that  $0 < \nu \leq 2\mu$ .

Therefore it seems possible that for operator  $A_1$  there exists a Lipschitz constant  $q'$  with  $q' < a$ .

If  $C(u, v)$  is of the form (6) and there exists an  $\omega > 0$  such that we have

$$\omega\|u\| < \|H(v)u\|$$

for any  $u, v \in S$ , then

$$\omega - rL \leq \nu \leq h + rL = q.$$

Indeed, we have

$$\begin{aligned} \nu\|u - u'\| &\geq \|H(u)u - H(u')u'\| \\ &= \|H(u)u - H(u')u + H(u')u - H(u')u'\| \\ &\geq \|H(u')[u - u']\| - \|[H(u) - H(u')]u\| \\ &\geq \omega\|u - u'\| - rL\|u - u'\| = (\omega - rL)\|u - u'\|. \end{aligned}$$

for any  $u, u' \in S$ .

Now we can formulate the following

**THEOREM 5.** *If the assumptions of Theorem 4 are satisfied and the inequality*

$$(22) \quad \|f\| < \frac{\omega(1-h)^2}{L(2-h)}$$

*holds true, then any Lipschitz constant  $\nu$  of  $A_1$  satisfies the inequality  $\alpha < \nu$  and thus the error estimation (19) is better than error estimation (20) with  $q$  replaced by  $\nu$ .*

**Proof.** From (22) we obtain successively

$$\begin{aligned} L\|f\|(2-h) &< \omega(1-h)^2, \\ L\|f\| + L\|f\|(1-h) &< \omega(1-h)^2, \\ \alpha = \frac{L\|f\|}{(1-h)^2} &< \omega - \frac{L\|f\|}{1-h} = \omega - Lr \leq \nu \leq q < 1, \end{aligned}$$

which proves Theorem 5.

Consider now a simple example. Let  $B$  be the real line. Then  $H(u)$  is a continuous function. If, moreover, we have

$$0 < \omega < |H(u)| < h < 1,$$

then for any  $f$  with

$$|f| < \frac{\omega(1-h)^2}{L(2-h)}$$

the equation

$$u = f + H(u)u$$

has a unique solution which may be obtained by either PM or MPM, and the error estimation for MPM is better than error estimation for PM with any possible  $\nu$ .

More specifically, if

$$H(u) = \frac{1}{2} + \frac{1}{4} \sin 10^{-6}u,$$

then we have

$$h = \frac{3}{4}, \quad \omega = \frac{1}{4}, \quad L = 10^{-6} \quad \text{and} \quad |f| < \frac{1}{8} 10^5.$$

If  $f = 10^2$ , then  $\alpha = 1,6 \cdot 10^{-3}$  and the error estimation for MPM takes the form

$$|u - \tilde{u}_n| \leq (1,6 \cdot 10^{-3})^n \cdot 4 \cdot 10^2 = (0,4)^{2n+1} \cdot 10^{-2n+3}.$$

On the other hand, we have

$$0,2496 = \frac{1}{4} - 10^{-6} \cdot 4 \cdot 10^2 \leq \nu \leq \frac{3}{4} + 10^{-6} \cdot 4 \cdot 10^2 = 0,7504,$$

and, therefore, the error estimation for PM is

$$|u - u_n| \leq v^n \cdot 4 \cdot 10^2,$$

which is evidently worse than the former.

4. Now we shall consider the more general equation

$$(23) \quad u = Au + H(u)u + f,$$

where operator  $A$  satisfies (2) and operator  $H(u)$  is linear and has properties (7) and (8).

Equation (23) includes both equation (1) and equation (14). Indeed, if  $Hu \equiv \Theta$ , we get (1) and if  $Au \equiv \Theta$ , we get (14).

As a generalization of Lemma A, we have

LEMMA B. *If (2), (7), (10) and the inequality  $q + h < 1$  hold true,  $A\Theta = \Theta$  and  $u, u'$  are solutions of equations*

$$(24) \quad u = Av + H(v)u + f, \quad u' = Av' + H(v')u' + f',$$

then we have the estimations

$$(25) \quad \|u\| \leq \frac{\|f\|}{1 - (q + h)}, \quad \|u'\| \leq \frac{\|f'\|}{1 - (q + h)},$$

$$(26) \quad \|u - u'\| \leq \frac{\|f - f'\|}{1 - h} + \frac{q(1 - h - q) + \min(\|f\|, \|f'\|)L}{(1 - h)(1 - h - q)} \|v - v'\|.$$

Proof. Estimations (25) follows immediately from (4). Subtracting the second equation (24) from the first one we get

$$\begin{aligned} \|u - u'\| &\leq \|f - f'\| + q\|v - v'\| + h\|u - u'\| + \|u'\|L\|v - v'\|, \\ \|u - u'\| &\leq \frac{\|f - f'\|}{1 - h} + \frac{q(1 - h - q) + \|f'\|L}{(1 - h)(1 - h - q)} \|v - v'\|. \end{aligned}$$

Likewise we get the inequality

$$\|u - u'\| \leq \frac{\|f - f'\|}{1 - h} + \frac{q(1 - h - q) + \|f\|L}{(1 - h)(1 - h - q)} \|v - v'\|.$$

The last two inequalities yield estimation (26).

It is evident that if  $Au \equiv \Theta$ , then Lemma A results from Lemma B, because in this case  $q = 0$ .

We can now formulate a theorem which is a generalization both of Theorem 1 and Theorem 2.

THEOREM 6. *If assumptions (2), (7), (8) are fulfilled, where*

$$\alpha' = \frac{q(1 - h - q) + \|f\|L}{(1 - h)(1 - h - q)} < 1$$

and there exists a  $u_0 \in D$  such that  $u_n \in D$  for any  $n = 0, 1, \dots$  where the  $u_n$ 's are defined recursively by equations

$$u_{n+1} = Au_n + H(u_n)u_{n+1} + f,$$

then equation (23) has a solution  $u$  which is unique in  $D$ , solution  $u$  is the limit of the sequence  $\{u_n\}$  defined above and we have the estimations

$$\|u\| \leq \frac{\|f\|}{1-h-q}, \quad \|u-f\| \leq \frac{(q+h)\|f\|}{1-h-q},$$

$$\|u-u_n\| \leq (\alpha')^n \|u-u_0\| \leq (\alpha')^n \left[ \|f-u_0\| + \frac{(q+h)\|f\|}{1-h-q} \right], \quad n = 0, 1, \dots$$

**Proof.** In order to prove this theorem it is sufficient to apply Lemma B.

It is easy to see that if  $H(u) \equiv \Theta$ , i.e. if  $h = 0$  and  $L = 0$ , then Theorem 1 follows from Theorem 6. Let us note that in this case there are no restrictions concerning  $\|f\|$ .

If  $Au \equiv \Theta$ , i.e.  $q = 0$ , then we obtain Theorem 2 from Theorem 6.

**5.** We shall now apply the results obtained above to integral equations. Let us consider the Banach space  $B(G)$  of continuous functions with values in the Banach space  $B_1$ , defined over a bounded set  $G \subset R^p$ , where  $R^p$  denotes the  $p$ -dimensional Euclidean space. The norm of  $u(x) \in B(G)$ , where  $x = (x_1, \dots, x_p)$ , is defined by

$$\|u(x)\| = \sup_{x \in G} \|u(x)\|_1,$$

where  $\|\cdot\|_1$  denotes the norm in the space  $B_1$ .

Let  $K(x, y, u)$  be a non-linear, continuous operator, defined for  $(x, y, u) \in G \times G \times D_1$  with values in  $B_1$ , where  $D_1$  is some closed subset of  $B_1$ .

Consider now the equation

$$(27) \quad u(x) = \int_G K(x, y, u(y)) dy + f(x),$$

where  $f(x) \in B(G)$ .

**CONCLUSION 1.** *If we have*

$$\|K(x, y, u) - K(x, y, \bar{u})\|_1 \leq A \|u - \bar{u}\|_1, \quad A \text{mes} G < 1,$$

and there exists a  $u_0(y)$  such that  $u_0(y) \in D_1$  for any  $y \in G$  and for any  $y \in G$ , all PM iterations  $u_n$ ,  $n = 1, 2, \dots$ , belong to  $D_1$ , then equation (27) has a unique solution  $u$  and the sequence  $\{u_n\}$  converges uniformly to  $u$ .

**CONCLUSION 2.** *If the operator  $K(x, y, u)$  is of the form*

$$K(x, y, u) = H(x, y, u) \cdot u,$$



where  $H(x, y, v)$  is a linear operator satisfying for any  $(x, y, v), (x, y, \bar{v}) \in G \times G \times D_1$  the inequalities

$$\begin{aligned} \|H(x, y, v)\| &\leq h, \\ \|H(x, y, v) - H(x, y, \bar{v})\| &\leq L\|v - \bar{v}\|_1, \\ h \operatorname{mes} G &< 1, \quad L\|f\| \operatorname{mes} G < (1 - h \operatorname{mes} G)^2, \end{aligned}$$

and if there exists  $u_0(y) \in D_1$  such that all PM and MPM iterations belong to  $D_1$  for  $x \in G$ , then equation (27) has a unique solution  $u(x)$  and the sequences  $\{u_n(x)\}$  and  $\{\tilde{u}_n(x)\}$  defined by relations

$$\begin{aligned} u_{n+1}(x) &= \int_G H(x, y, u_n(y)) u_{n+1}(y) dy + f(x), \\ \tilde{u}_{n+1}(x) &= \int_G H(x, y, \tilde{u}_n(y)) \tilde{u}_n(y) dy + f(x), \\ \tilde{u}_0(x) &= u_0(x), \quad n = 0, 1, \dots, \end{aligned}$$

converge uniformly to  $u(x)$ .

Conclusion 2 implies that if we take integral equation considered by Picone [6] (p. 124) of the form (27) with  $p = 1$  (now  $B_1$  is the real line), and

$$K(x, y, u) = \begin{cases} \varphi(x, y) u \sin\left(\frac{\pi}{2} - 1 + u^\gamma\right) & \text{for } |u| \geq 1, \\ \varphi(x, y) u^n & \text{for } |u| < 1, \end{cases}$$

where  $\varphi(x, y)$  is continuous and if we have  $|\varphi(x, y)| < \mu$  for  $(x, y) \in G \times G$ , then the sequence of PM iterations is convergent if

$$\gamma \leq 0, \quad \operatorname{mes} G < \frac{1}{\mu \max(n, 1 + |\gamma|)}$$

and  $f(x)$  is any continuous function, or if

$$\gamma \leq 1, \quad \operatorname{mes} G < \frac{1}{\mu},$$

and the function  $f(x)$  satisfies the condition

$$\sup_{x \in G} |f(x)| < \frac{(1 - \mu \operatorname{mes} G)^2}{\max(n - 1, |\gamma|) \operatorname{mes} G};$$

in the latter case the sequence of MPM iterations is also convergent.

**6.** Now, let the functions  $K(x, y, u), f(x)$  and the linear operator  $H(x, y, v)u$  be defined and continuous for  $(x, y, u), (x, y, v) \in R_+^1 \times R_x^1 \times B_1$ , with values in  $B_1$ , where  $R_+^1 = \langle x_0, +\infty \rangle$ ,  $R_x^1 = \langle x_0, x \rangle$ ,  $B_1$  is any Banach space and  $K(x, y, \Theta) \equiv \Theta$ .

Let us assume that for any  $\tau \in R_+^1$  there exist functions  $\lambda_\tau(y)$ ,  $h_\tau(y)$ ,  $L_\tau(y)$  continuous for  $y \in R_\tau^1$  such that for any  $(x, y, u), (x, y, \bar{u}) \in R_+^1 \times R_x^1 \times B_1$  the inequalities

$$(28) \quad \|K(x, y, u) - K(x, y, \bar{u})\|_1 \leq \lambda_\tau(y) \|u - \bar{u}\|_1,$$

$$(29) \quad \|H(x, y, u)\| \leq h_\tau(y)$$

are satisfied and the inequality

$$(30) \quad \|H(x, y, u(y)) - H(x, y, \bar{u}(y))\| \leq L_\tau(y) \|u(y) - \bar{u}(y)\|_1$$

holds for  $(x, y) \in R_\tau^1 \times R_x^1$  and  $u(x), \bar{u}(x) \in S'$ , where

$$S': \quad \|u(x)\|_1 \leq \sup_{0 \leq s \leq \tau} \|f(s)\|_1 \exp \left[ \int_{x_0}^x (h_\tau(y) + \lambda_\tau(y)) dy \right].$$

We shall consider Volterra's integral equation

$$(31) \quad u(x) = f(x) + \int_{x_0}^x K(x, y, u(y)) dy + \int_{x_0}^x H(x, y, u(y)) u(y) dy.$$

For any fixed  $\tau \in R_+^1$ , let us consider the Banach space  $B(R_\tau^1)$  of continuous functions defined for  $x \in R_\tau^1$  with the norm

$$_\tau \|u(x)\| = \sup_{x \in R_\tau^1} \|u(x)\|_1$$

and the subset  $T \in B(R_\tau^1)$  of those  $u(x)$  which fulfils the inequality

$$\|u(x)\|_1 \leq _\tau \|f(x)\| \exp \left[ \int_{x_0}^x (h_\tau(y) + \lambda_\tau(y)) dy \right] = \gamma_{1\tau}(x).$$

Denote by  $\Omega_1$  the operator standing at the right-hand side of equation (31). It is easy to see that  $\Omega_1$  maps  $T$  into  $T$ . Indeed, if  $u \in T$  and

$$w(x) = \int_{x_0}^x K(x, y, u(y)) dy + \int_{x_0}^x H(x, y, u(y)) u(y) dy + f(x),$$

then

$$\begin{aligned} \|w(x)\|_1 &\leq _\tau \|f(x)\| + \int_{x_0}^x [h_\tau(y) + \lambda_\tau(y)] _\tau \|f(x)\| \exp \left( \int_{x_0}^y [h_\tau(s) + \lambda_\tau(s)] ds \right) dy \\ &\leq _\tau \|f(x)\| \exp \left( \int_{x_0}^x [h_\tau(y) + \lambda_\tau(y)] dy \right). \end{aligned}$$

For any  $u(x)$  and  $\bar{u}(x) \in T$  we get

$$(32) \quad \|w(x) - \bar{w}(x)\|_1 \leq \int_{x_0}^x [\lambda_\tau(y) + h_\tau(y) + L_\tau(y) \gamma_{1\tau}(y)] \|u(y) - \bar{u}(y)\| dy.$$

Let

$$\lambda_{1\tau}(y) = h_{\tau}(y) + \lambda_{\tau}(y) + \gamma_{1\tau}(y) L_{\tau}(y).$$

If we now define a new norm in the space  $B(R_{\tau}^1)$  putting

$$[u(x)]_{1\tau} = \sup_{x_0 \leq x \leq \tau} \left[ \|u(x)\|_1 \exp \left( -\frac{1}{q} \int_{x_0}^x \lambda_{1\tau}(y) dy \right) \right],$$

where  $q < 1$  is an arbitrary positive constant, we can easily show that  $\Omega_1$  has the contraction property in  $T$ , i.e. it satisfies in  $T$  a Lipschitz condition with the constant  $q < 1$ .

In fact, from (32) we get

$$\begin{aligned} \|w(x) - \bar{w}(x)\|_1 &\leq [u(x) - \bar{u}(x)]_{1\tau} \cdot q \int_{x_0}^x \frac{1}{q} \lambda_{1\tau}(y) \exp \left( \frac{1}{q} \int_{x_0}^y \lambda_{1\tau}(s) ds \right) dy \\ &= [u(x) - \bar{u}(x)]_{1\tau} q \left[ \exp \left( \frac{1}{q} \int_{x_0}^x \lambda_{1\tau}(s) ds \right) - 1 \right] \\ &\leq q [u(x) - \bar{u}(x)]_{1\tau} \exp \left( \frac{1}{q} \int_{x_0}^x \lambda_{1\tau}(s) ds \right), \end{aligned}$$

which yields

$$(33) \quad [w(x) - \bar{w}(x)]_{1\tau} \leq q [u(x) - \bar{u}(x)]_{1\tau}.$$

**THEOREM 7.** *If assumptions (28), (29) and (30) are satisfied, then equation (31) has a unique solution  $u(x)$  defined for  $x \in R_+^1$  and belonging to  $T$ , the sequence  $\{\tilde{u}_n(x)\}$  of PM iterations defined by  $\tilde{u}_0(x) \equiv 0$  and*

$$(34) \quad \tilde{u}_{n+1}(x) = f(x) + \int_{x_0}^x K(x, y, \tilde{u}_n(y)) dy + \int_{x_0}^x H(x, y, \tilde{u}_n(y)) \tilde{u}_n(y) dy$$

*is uniformly convergent in  $R_{\tau}^1$  to  $u(x)$  for any  $\tau \in R_+^1$  and we have the estimations*

$$\|u(x) - \tilde{u}_n(x)\|_1 \leq \frac{1}{n!} \left[ \int_{x_0}^x \lambda_{1\tau}(s) ds \right]_{\tau}^n \|f(x)\| \exp \left( \int_{x_0}^x [h_{\tau}(y) + \lambda_{\tau}(y)] dy \right)$$

for  $x \in R_{\tau}^1$ , or

$$[u(x) - \tilde{u}_n(x)]_{1\tau} \leq \frac{q^n}{1-q} [f(x)]_{1\tau}$$

for  $n = 0, 1, \dots$

**Proof.** In order to prove this theorem it is sufficient to apply (33), (34) and Theorem 1.

7. Now we shall discuss the problem of convergence of the sequence of MPM iterations  $u_n(x)$  constructed in the following manner:  $u_0(x) \equiv \theta$  and if  $u_n(x)$  is given, then  $u_{n+1}(x)$  is obtained from the equation

$$(35) \quad u_{n+1}(x) = f(x) + \int_{x_0}^x K(x, y, u_n(y)) dy + \int_{x_0}^x H(x, y, u_n(y)) u_{n+1}(y) dy.$$

Let us consider the operator  $\Omega_2$  which transforms any function  $u(x) \in B(R_\tau^1)$  on a function  $v(x)$  which is a solution of the equation

$$(36) \quad v(x) = f(x) + \int_{x_0}^x K(x, y, u(y)) dy + \int_{x_0}^x H(x, y, u(y)) v(y) dy.$$

At first, we shall show that there exist functions  $\gamma_{2\tau}(x)$  and  $\omega_\tau(x)$  such that the set  $T_1 \subset B(R_\tau^1)$  of those  $u(x)$  which fulfil the inequality

$$(37) \quad \|u(x)\|_1 \leq \gamma_{2\tau}(x) \leq \omega_\tau(x) \exp\left(\int_{x_0}^x \omega_\tau(s) ds\right)$$

is transformed by  $\Omega_2$  into  $T_1$ .

Indeed, from (36) we have

$$\|v(x)\|_1 \leq \|f(x)\|_1 + \int_{x_0}^x \lambda_\tau(y) \gamma_{2\tau}(y) dy + \int_{x_0}^x h_\tau(y) \|v(y)\|_1 dy.$$

Further, using well known Gronwall's Lemma ([1], chap. 10) we get

$$\begin{aligned} \|v(x)\|_1 &\leq \|f(x)\|_1 + \int_{x_0}^x \lambda_\tau(y) \gamma_{2\tau}(y) dy + \\ &+ \int_{x_0}^x \left[ \|f(y)\|_1 + \int_{x_0}^y \lambda_\tau(s) \gamma_{2\tau}(s) ds \right] \cdot \left[ h_\tau(y) \exp \int_y^x h_\tau(s) ds \right] dy. \end{aligned}$$

In order to prove the relation  $\Omega_2(T_1) \subset T_1$  it is sufficient to take for  $\gamma_{2\tau}(x)$  a solution of the equation

$$\begin{aligned} z(x) &= \|f(x)\|_1 + \int_{x_0}^x \lambda_\tau(y) z(y) dy + \\ &+ \int_{x_0}^x \left[ \|f(y)\|_1 + \int_{x_0}^y \lambda_\tau(s) z(s) ds \right] \left[ h_\tau(y) \exp \int_y^x h_\tau(s) ds \right] dy. \end{aligned}$$

It can be proved that function  $\gamma_{2\tau}(x)$  defined in a such manner satisfies (37) with

$$\begin{aligned} \omega_\tau(x) &= \sup_{x_0 \leq y \leq x} \max[\|f(y)\|_1, \lambda_\tau(y)] \times \\ &\times \left[ 1 + \max(1; \tau) \sup_{x_0 \leq y \leq x} h_\tau(y) \exp \int_{x_0}^y h_\tau(s) ds \right]. \end{aligned}$$

Now, we shall show that  $\Omega_2$  has the contraction property in  $T_1$ . Let  $u(x), \bar{u}(x) \in T_1$  and  $v(x), \bar{v}(x)$  be values of  $\Omega_2$  for  $u(x)$  and  $\bar{u}(x)$ , respectively.

We have

$$\begin{aligned} \|v(x) - \bar{v}(x)\|_1 &\leq \int_{x_0}^x \lambda_\tau(y) \|u(y) - \bar{u}(y)\|_1 dy + \\ &\quad + \int_{x_0}^x \|H(x, y, u(y))v(y) - H(x, y, \bar{u}(y))\bar{v}(y)\|_1 dy \\ &\leq \int_{x_0}^x \lambda_\tau(y) \|u(y) - \bar{u}(y)\|_1 dy + \int_{x_0}^x \gamma_{2\tau}(y) L_\tau(y) \|u(y) - \bar{u}(y)\|_1 dy + \\ &\quad + \int_{x_0}^x h_\tau(y) \|v(y) - \bar{v}(y)\|_1 dy. \end{aligned}$$

From Gronwall's Lemma we get again

$$\begin{aligned} \|v(x) - \bar{v}(x)\|_1 &\leq \int_{x_0}^x [\lambda_\tau(y) + \gamma_{2\tau}(y) L_\tau(y)] \|u(y) - \bar{u}(y)\|_1 dy + \\ &\quad + \int_{x_0}^x \left\{ \left[ \int_{x_0}^y [\lambda_\tau(s) + \gamma_{2\tau}(s) L_\tau(s)] \|u(s) - \bar{u}(s)\|_1 ds \right] h_\tau(y) \exp \left( \int_y^x h_\tau(s) ds \right) \right\} dy \end{aligned}$$

and

$$\begin{aligned} \sup_{x_0 \leq s \leq x} \|v(s) - \bar{v}(s)\|_1 &\leq \int_{x_0}^x \{ [\lambda_\tau(y) + \gamma_{2\tau}(y) L_\tau(y)] + \\ &\quad + \left[ \int_{x_0}^y (\lambda_\tau(s) + \gamma_{2\tau}(s) L_\tau(s)) ds \right] h_\tau(y) \exp \left( \int_y^\tau h_\tau(s) ds \right) \} \sup_{x_0 \leq s \leq y} \|u(s) - \bar{u}(s)\|_1 dy. \end{aligned}$$

Put

$$\lambda_{2\tau}(y) = \lambda_\tau(y) + \gamma_{2\tau}(y) L_\tau(y) + \left[ \int_{x_0}^y [\lambda_\tau(s) + \gamma_{2\tau}(s) L_\tau(s)] ds \right] h_\tau(y) \exp \left( \int_y^\tau h_\tau(s) ds \right).$$

The last inequality can now be written in the form

$$\sup_{x_0 \leq s \leq x} \|v(s) - \bar{v}(s)\|_1 \leq \int_{x_0}^x \lambda_{2\tau}(y) \sup_{x_0 \leq s \leq y} \|u(s) - \bar{u}(s)\|_1 dy.$$

If we now define a new norm in the space  $B(R_\tau^1)$  by

$$[u(x)]_{2\tau} = \sup_{x_0 \leq x \leq \tau} \left[ \sup_{x_0 \leq s \leq x} \|u(s)\|_1 \exp \left( -\frac{1}{q} \int_{x_0}^x \lambda_{2\tau}(y) dy \right) \right],$$

where  $0 < q < 1$ , then by an argument similar to that above we get

$$(38) \quad [v(x) - \bar{v}(x)]_{2\tau} \leq q [u(x) - \bar{u}(x)]_{2\tau}.$$

THEOREM 8. *Under the assumptions of Theorem 7 the sequence of MPM iterations defined by (35) is uniformly convergent in  $R_\tau^1$  and error estimation*

$$\sup_{x_0 \leq s \leq x} \|u(s) - u_n(s)\|_1 \leq \frac{1}{n!} \left[ \int_{x_0}^x \lambda_{2\tau}(y) dy \right]^n \gamma_{2\tau}(x), \quad x \in R_\tau^1,$$

or

$$[u(x) - u_n(x)]_{2\tau} \leq \frac{q^n}{1-q} [f(x)]_{2\tau}$$

holds for any  $\tau \in R_+^1$  and  $n = 0, 1, \dots$

Proof. In order to prove this theorem it is sufficient to apply (38) and Theorem 1.

Let us observe that  $\lambda_{1\tau}(y)$  is not, in general, the smallest Lipschitz coefficient for the operator

$$(39) \quad K(x, y, u) + H(x, y, u)u.$$

Suppose that there exists a continuous function  $\omega_\tau(y)$  such that

$$(40) \quad \|H(x, y, u)u\|_1 > \omega_\tau(y) \|u\|_1,$$

for  $x \in R_\tau^1$ ,  $y \in R_x^1$  and  $u \in B_1$ . Then we get

$$\begin{aligned} \|K(x, y, u) + H(x, y, u)u - K(x, y, \bar{u}) - H(x, y, \bar{u})\bar{u}\|_1 \\ \geq [\omega_\tau(y) - \lambda_\tau(y) - L_\tau(y)\gamma_{1\tau}(y)] \|u - \bar{u}\|_1. \end{aligned}$$

From here we infer that any Lipschitz coefficient for (39), say  $v_\tau(y)$ , satisfies the inequality

$$[\omega_\tau(y) - \lambda_\tau(y) - L_\tau(y)\gamma_{1\tau}(y)] \leq v_\tau(y) \leq \lambda_{1\tau}(y).$$

Thus we can formulate

THEOREM 9. *If the assumptions of Theorem 7, (40) and*

$$(41) \quad \lambda_{2\tau}(y) < \omega_\tau(y) - \lambda_\tau(y) - \gamma_{1\tau}(y)L_\tau(y)$$

*are fulfilled, then error estimation for MPM is better than that for PM.*

If  $K(x, y, u) \equiv \theta$ , then

$$\gamma_{2\tau}(y) = \gamma_{1\tau}(y) = {}_\tau\|f(x)\| \exp\left(\int_{x_0}^x h_\tau(s) ds\right)$$

and (41) takes the form

$$\begin{aligned} 2_\tau\|f(x)\| L_\tau(y) \exp\left(\int_{x_0}^y h_\tau(s) ds\right) + \\ + \int_{x_0}^y \left[ {}_\tau\|f(x)\| L_\tau(s) \exp\left(\int_{x_0}^s h_\tau(\sigma) d\sigma\right) ds \right] h_\tau(y) \exp\left(\int_y^\tau h_\tau(s) ds\right) < \omega_\tau(y). \end{aligned}$$

Hence it is evident that the last inequality can be satisfied if  ${}_{\tau}\|f(x)\| \times L_{\tau}(y)$  is sufficiently small.

In general, the inequality

$$(42) \quad \lambda_{2\tau}(y) < \lambda_{1\tau}(y)$$

can be fulfilled if  $\lambda_{\tau}(y)$  and  ${}_{\tau}\|f(x)\| L_{\tau}(y)$  are sufficiently small.

If  $K(x, y; u) \equiv \theta$ , then (42) can be written in the form

$$\left[ \int_{x_0}^y {}_{\tau}\|f(x)\| \exp\left(\int_0^s h_{\tau}(\sigma) d\sigma\right) L_{\tau}(s) ds \right] \exp\left(\int_y^{\tau} h_{\tau}(s) ds\right) < 1.$$

For example, let us take the integral equation

$$u(x) = f(x) + \int_0^x a(x, y) [2 + \sin 10^{-6} u(y)] u(y) dy,$$

where  $f(x)$  and  $a(x, y)$  are continuous for  $x \in \langle 0, \tau \rangle$  and  $(x, y) \in \langle 0, \tau \rangle \times \langle 0, x \rangle$ , respectively.

Put

$$\delta(y) = \sup_{x \in \langle 0, \tau \rangle} |a(x, y)|, \quad F = \sup_{0 \leq x \leq \tau} |f(x)|.$$

Then

$$h_{\tau}(y) = 3\delta(y), \quad \gamma_{1\tau}(y) = F \exp\left(3 \int_0^y \delta(s) ds\right), \quad L_{\tau}(y) = 10^{-6},$$

$$\lambda_{1\tau}(y) = 3\delta(y) + 10^{-6} F \exp\left(3 \int_0^y \delta(s) ds\right),$$

$$\begin{aligned} \lambda_{2\tau}(y) = & 10^{-6} F \exp\left(3 \int_0^y \delta(s) ds\right) + \\ & + \left[ \int_0^y 10^{-6} F \exp\left(3 \int_0^s \delta(\sigma) d\sigma\right) ds \right] 3\delta(y) \exp\left(3 \int_y^{\tau} \delta(s) ds\right), \end{aligned}$$

$$\omega_{\tau}(y) = \inf_{0 \leq x \leq \tau} |a(x, y)|.$$

In this case inequality (42) is satisfied for

$$F < 10^6 \left[ \tau \exp\left(6 \int_0^{\tau} \delta(y) dy\right) \right]^{-1},$$

whereas inequality (41) holds for

$$\begin{aligned} F < & 10^6 \exp\left(-3 \int_0^{\tau} \delta(y) dy\right) \times \\ & \times \left[ 2 + 3\tau \sup_{0 \leq y \leq \tau} \delta(y) \exp\left(3 \int_0^{\tau} \delta(y) dy\right) \right]^{-1} \inf_{0 \leq y \leq x \leq \tau} |a(x, y)|. \end{aligned}$$

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Reçu par la Rédaction le 31. 3. 1965