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ON A SYSTEM OF INTEGRAL INEQUALITIES

BY

C. OLECH (KRAKÓW)

The present note is concerned with a system of integral inequalities

$$\varphi_i(t) \leqslant c_i + \int_{t_0}^{t} f_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)d\tau, \qquad i = 1, \ldots, n, \tag{1}$$

$t \in [t_0, t_1) = J$, $t_1 \leqslant +\infty$.

For simplicity, we introduce the relation "$\leqslant$" in R^n, namely we set for any two points of R^n, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

$$x \leqslant y \text{ iff } x_i \leqslant y_i \quad \text{for each } i = 1, \ldots, n. \tag{2}$$

Relation (2) is a semi-order in R^n and it is easy to see that for any bounded set $A \subset R^n$ there exists the sup A with respect to relation (2), which is

$$\sup A = \min\{z \in R^n: x \leqslant z \text{ for each } x \leqslant A\}. \tag{3}$$

We shall need (3) only for two point sets. In that case we have

$$\sup\{x, y\} = z = (z_1, \ldots, z_n) \quad \text{where} \quad z_i = \max(x_i, y_i) \tag{4}$$

(x_i and y_i are coordinates of x and y respectively).

We can now write (1) in a shorter form, namely

$$\varphi(t) \leqslant c + \int_{t_0}^{t} f\big(\tau, \varphi(\tau)\big)d\tau, \qquad t \in J, \tag{5}$$

where φ, c and f are now n-vectors.

The purpose of this note is to present a simple and very short proof of a result due to Opial [4] concerning inequality (5) (Theorem A below). The proof is based on an idea used by Cafiero [1, 2] to prove an analogous result on differential inequality (cf. also [3]).

THEOREM A (Opial). *Let a map $f: J \times R^n \to R^n$ be continuous and assume that*

$$f(t, x) \leqslant f(t, y) \quad \textit{for any } x \leqslant y. \tag{6}$$

If a continuous map $\varphi: J \to R^n$ *satisfies inequality* (5) *and* $\psi: [t_0, t_2) \to R^n$ $(t_2 \leqslant t_1)$ *is the maximal solution of*

$$x(t) = c + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau, \tag{7}$$

then

$$\varphi(t) \leqslant \psi(t) \quad \textit{for} \quad t_0 \leqslant t < t_2. \tag{8}$$

(A solution $\psi(t)$ is the *maximal solution* of (7) if for any other solution $x(t)$ of (7) the inequality $x(t) \leqslant \psi(t)$ holds on the common interval of existence; if (6) holds, then the maximal solution of (7) exists, cf. [5].)

Proof. Put

$$F(t, x) = f(t, \sup\{x, \varphi(t)\}). \tag{9}$$

By (4), $\varphi(t) \leqslant \sup\{x, \varphi(t)\}$; therefore from (6) and (9) we have

$$F(t, x) \geqslant f(t, \varphi(t)) \quad \text{for each } x. \tag{10}$$

Let $\chi: [t_0, t_2) \to R^n$ be the maximal solution of

$$x(t) = c + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau. \tag{11}$$

Then, using (10) and (5), we get

$$\chi(t) = c + \int_{t_0}^{t} F(\tau, \chi(\tau))\, d\tau \geqslant c + \int_{t_0}^{t} f(\tau, \varphi(\tau))\, d\tau \geqslant \varphi(t). \tag{12}$$

It follows from (12) and (4) that $\sup\{\chi(t), \varphi(t)\} = \chi(t)$. Therefore by (9) we have $F(t, \chi(t)) = f(t, \chi(t))$, whence $\chi(t)$ is also the maximal solution of (7). Thus (12) proves (8) and completes the proof of Theorem A.

Remarks. The monotonicity assumption (6) is essential for Theorem A (cf. [4]).

The corresponding result concerning a system of differential inequalities [5] requires that $f_i(t, x_1, \ldots, x_n)$ is non-decreasing with respect to x_j only for $j \neq i$. In that sense differential inequalities theory is more general. In fact, one can reduce a proof of Theorem A to a corresponding result on differential inequalities (cf. [6]).

A special case of Theorem A, namely $n = 1$ and f linear in x, is the celebrated Gronwall's lemma.

REFERENCES

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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