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ON COMPACT HAUSDORFF SPACES HAVING FINITELY MANY TYPES OF OPEN SUBSETS

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Herein, a *continuum* is a compact connected Hausdorff space, βX is the Čech-Stone compactification of a completely regular space X, and X^* is the remainder $\beta X - X$ of the space βX.

A point p of a space X is said to be a *local separating point* of X (shortly, l.s.-point of X) if there exists a neighbourhood U of p such that $U - \{p\} = W_1 \cup W_2$, where W_1 and W_2 are open, disjoint and have non-empty intersections with the component of U containing p.

Schoenfeld and Gruenhage [5] showed that the Cantor set is the unique infinite compact metric space having only two topologically distinct non-empty open subsets. In Section 1 of the present paper we show that the Cantor set is the unique dense in itself compact metric space having only finite number of topologically distinct open subsets or, as we say shortly, having finitely many types of open subsets. This implies that every non-degenerate metric continuum has infinitely many types of open subsets and leads to the problem of the existence of non-metric continua having only finitely many types of open subsets.

However, it seems to be interesting to solve first the problem of the existence of non-metric continua having exactly three types of (non-empty) open subsets. In Section 2 we prove that such continua are perfectly normal. By a theorem of Juhász [2], there exists no non-separable perfectly normal compact Hausdorff space in the theory ZFC+Martin's axiom + negation of the Continuum Hypothesis. Hence, only possible in ZFC examples of continua having only three types of open subsets are the separable ones. However, we do not know of any such an example. **(P 1072)**

The class of compact Hausdorff spaces having at most three types of open subsets is much more wider than that having two types only: these are dense in itself, totally disconnected and perfectly normal (see [4]). As was noted by R. Frankiewicz, there exist non-perfectly normal compact

Hausdorff spaces having three types of open subsets; the example of such a space is given in Section 3.

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1. A characterization of the Cantor set. The following lemma is a particular case of theorem by Whyburn [6] (Theorem 9.2, p. 61).

1.1. *Let G be a set of all l.s.-points of a locally compact separable metric space X. There exists a countable set F such that if $p \in G-F$, then p has a base consisting of open sets having only two points on the boundary.*

COROLLARY. *Every continuum of convergence of a locally compact separable metric space X contains at most countably many l.s.-points of X.*

1.2. *Let X be a continuum. If the set $(X-\{p\})^*$ is disconnected, then p is an l.s.-point of X.*

Proof. Let $(X-\{p\})^* = F_1 \cup F_2$, where F_1 and F_2 are non-empty, compact and disjoint. Let W_1' and W_2' be open subsets of $\beta(X-\{p\})$ such that $F_i \subset W_i'$ and $\operatorname{cl} W_1 \cap \operatorname{cl} W_2 = \emptyset$. Shrink each of F_i to a point and call q the quotient map. The space $q(\beta(X-\{p\}))$ is a continuum if p does not cut X or the union of two continua in the opposite case. Thus, by Janiszewski's lemma ([3], Theorem 2, p. 172), the component of $q(W_i')$ containing the point $q(F_i)$ is non-degenerate. Now, shrink the points $q(F_1)$ and $q(F_2)$ to a point and call r the quotient map. The space $rq(\beta(X-\{p\}))$ is homeomorphic to X and the point in $r(q(F_1) \cup q(F_2))$ may be identified with p. So, if we put

$$U = rq(W_1' \cup W_2') \quad \text{and} \quad W_i = rq(W_i' - F_i),$$

then according to the definition of an l.s.-point one can see that p is such a point of X.

COROLLARY. *If $p_1, \ldots, p_n$, $p_i \neq p_j$ for $i \neq j$, are non-l.s.-points of a continuum X, then the set $(X-\{p_1, \ldots, p_n\})^*$ has exactly n components.*

1.3. THEOREM. *If a dense in itself compact metric space X has only finite number of topologically distinct open subsets, then it is homeomorphic to the Cantor set.*

Proof. It suffices to show the total disconnectedness of X.

Suppose that X has a non-degenerate component and consider two cases.

I. There exists an infinite sequence $K_1, K_2, \ldots$ of distinct non-degenerate components of X. Let p_n be a non-cut point of K_n. Then the open sets $U_1, U_2, \ldots$, where $U_n = X-\{p_1, \ldots, p_n\}$, are topologically distinct, since U_n has exactly n non-compact components. A contradiction.

II. The sets $K_1, \ldots, K_n$ are all of non-degenerate components of X. Let U be a closed and open subset of X such that

$$K_1 \subset U \subset X-(K_2 \cup \ldots \cup K_n).$$

Consider two cases.

(i) The component K_1 is locally connected. Then for each positive integer k we can find a set U_k being open in K_1 and having exactly k components. Let W_k be an extension of U_k to an open subset of U. The set W_k has exactly k non-degenerate components, so the sets $W_1, W_2, \ldots$ are open and topologically distinct subsets of X. A contradiction.

(ii) The component K_1 is not locally connected. Then K_1 has a continuum of convergence and, by the Corollary to Lemma 1.1, there exists a sequence $p_1, p_2, \ldots$ of distinct non-l.s.-points of K_1. By the Corollary to 1.2, the sets $F_1, F_2, \ldots$, where $F_n = K_1 - \{p_1, \ldots, p_n\}$, are topologically distinct (since $F_1^*, F_2^*, \ldots$ are topologically distinct). Consequently, the sets $U_1, U_2, \ldots$, where $U_n = U-\{p_1, \ldots, p_n\}$, are topologically distinct, since F_n is a union of all non-degenerate components of U_n. A contradiction.

2. Continua having three types of open subsets. Let X be a continuum having three types of (non-empty) open subsets. One of these sets is X itself. Among non-compact sets there are connected and disconnected ones; the existence of connected ones follows from the generalization of Moore's theorem (see [1], Theorem 2-18, p. 49) about the existence of non-cut points in continua; the union of two disjoint open sets is an example of a disconnected open set.

2.1. *If U is a connected open subset of X, then U is dense in X.*

Proof. Assume that $X - \mathrm{cl}\, U \neq \emptyset$ and consider two cases.

(i) There exists an open component V of $X - \mathrm{cl}\, U$. Then, taking arbitrarily three disjoint open sets V_1, V_2, V_3, one can see that the set $V_1 \cup V_2 \cup V_3$ is open, disconnected and topologically different from the set $U \cup V$ which has only two components.

(ii) The set $X - \mathrm{cl}\, U$ has no open component. Then $X - \mathrm{cl}\, U$ is disconnected and topologically different from the disconnected open set $U \cup (X - \mathrm{cl}\, U)$ which has an open component, namely U.

2.2. *The continuum X is perfectly normal.*

Proof. At first we prove that every disconnected open subset of X is an F_σ-set. This will be shown if we construct a single disconnected open F_σ-set. Let F be a closed proper subset of X having a non-empty interior. Let U_1, $U_1 \neq X$, be an open neighbourhood of F. Take U_2, an open neighbourhood of F, contained with its closure in U_1. We get by induction a sequence $U_1 \supset \mathrm{cl}\, U_2 \supset U_2 \ldots$ of open neighbourhoods of F, the intersection of which is a proper closed subset E of X with a non-empty interior. Thus $X-E$ is an open F_σ-set and, by 2.1, $X-E$ is disconnected. Hence, all disconnected open subsets of X are F_σ-sets. To see that so are the connected ones, assume that U is one of them and take a closed set F having a non-empty interior and contained in U. The set $U-F$ is open and, by 2.1, is disconnected. Thus it is an F_σ-set. But $U = (U-F)\cup F$, so also U is an F_σ-set.

3. Example of a compact totally disconnected non-perfectly normal Hausdorff space having three types of open subsets (given to the author by R. Frankiewicz). Let D be a given set of cardinality $\aleph_1$ and let p_0 be a given point. Write

$$X_0 = \{p_0\} \quad \text{and} \quad X_n = X_{n-1}\times D \text{ for } n = 1, 2, \ldots$$

The set $X = \bigcup\{X_n \mid n = 0, 1, \ldots\}$ has a partial order. Namely, we assume $p < q$ iff there exists a sequence $\{q_0, \ldots, q_m\}$ such that $p = q_0$, $q = q_m$ and $q_i \in \{q_{i-1}\}\times D$. Let C be the set of all maximal chains in X. Put

$$Y = X\cup C$$

and

$$U(p) = \{q\in X \mid p \leqslant q\}\cup\{c\in C \mid p\in C\} \qquad \text{for all } p\in X.$$

We define topology on Y assuming that the sets $U(p)$ are closed and open in Y for all $p\in X$. Notice that Y is compact Hausdorff zero-dimensional space and each open subset of Y is homeomorphic either to Y or to the complement of the point $p\in X$, or to the complement of the point $c\in C$.

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