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# ON THE CONTINUITY OF CERTAIN NON-ADDITIVE SET FUNCTIONS

BY

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1. By no means non-additive set functions are something unusual in mathematics: outer measures, semi-variations of vector measures, capacities are widely known examples of such functions. However, most of the naturally arising non-additive set functions satisfy some subadditivity conditions which fairly well recompense the lack of additivity. On the other hand, they usually play only an auxiliary role in arguments concerning additive set functions (i.e. measures). Thus, for example, the use of variations, semi-variations, etc. in the theory of vector measures is comparable with that of semi-norms or quasi-norms in the theory of topological linear spaces and continuous linear mappings. This point of view is quite explicitly present, e.g., in the author's works (cf. [4]). Much attention is paid there to develop a theory of submeasures, i.e., subadditive monotone non-negative set functions on rings of sets. However, the chief motivation for this is that the submeasures are used as a convenient tool in investigating some properties of measures, especially those which can be expressed in terms of continuity with respect to the Fréchet-Nikodym (FN-) topologies of type $\Gamma(\mu)$.

In recent years several authors considered non-additive set functions, both real and group-valued, which were only assumed to be monotone or to satisfy conditions of Lipschitz type (see [1], [5] and the references given there, [3]).

Our main purpose is to indicate a method of investigating such functions, based upon a quite simple observation that they can be considered as functions continuous with respect to suitably chosen submeasures. In consequence, a number of results concerning non-additive set functions, originally proved by direct methods, can easily be derived from the theory of submeasures. We also answer in affirmative a problem posed by Dobrakov [3].

2. The following terminology and notation will be used without any further reference.

$\mathscr{R}$ denotes a ring of subsets of a set $T$. All set functions defined on $\mathscr{R}$ are assumed to vanish at $\emptyset$. $\boldsymbol{R}_+ = [0, \infty)$, and $\bar{\boldsymbol{R}}_+ = [0, \infty]$. A set function $\mu\colon \mathscr{R} \to \bar{\boldsymbol{R}}_+$ is said to be

*monotone* if $A \subset B$ implies $\mu(A) \leqslant \mu(B)$;

*subadditive* if $A \cap B = \emptyset$ implies $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$;

*exhaustive* if $\mu(E_n) \to 0$ for every (infinite) sequence $(E_n)$ of disjoint members of $\mathscr{R}$;

*order continuous* (at $\emptyset$) if $\mu(E_n) \to 0$ for every decreasing sequence $(E_n) \subset \mathscr{R}$ with empty intersection (we denote this by $E_n \searrow \emptyset$).

The notions of exhaustive and order continuous set functions with values in topological groups are defined similarly.

If $\eta\colon \mathscr{R} \to \bar{\boldsymbol{R}}_+$ is monotone and subadditive, it will be called a *submeasure* on $\mathscr{R}$ (see [4]). To each submeasure $\eta$ on $\mathscr{R}$ there corresponds an FN-topology $\Gamma(\eta)$, determined by the distance function (écart) $d_\eta$ on $\mathscr{R}$, defined by

$$d_\eta(A, B) = \eta(A \,\Delta\, B).$$

The reader is referred to [4] for a more detailed discussion of FN-topologies, submeasures, etc. We shall usually write shortly $(\mathscr{R}, \eta)$ instead of $(\mathscr{R}, \Gamma(\eta))$.

If $\mu$ is a set function on $\mathscr{R}$, we write $\mu \ll \eta$ to indicate that $\mu$ is continuous on $(\mathscr{R}, \eta)$ and then say that $\mu$ is *$\eta$-continuous*. Suppose that $\mu_1$ and $\mu_2$ are set functions on $\mathscr{R}$ determining FN-topologies $\Gamma(\mu_1)$ and $\Gamma(\mu_2)$, respectively; then we say that $\mu_1$ and $\mu_2$ are *equivalent* and write $\mu_1 \sim \mu_2$ if $\Gamma(\mu_1) = \Gamma(\mu_2)$.

**3.** In [3] Dobrakov has initiated a theory of set functions which we shall call $D$-submeasures (submeasures in [3]), intended to be — in his own words — "a non-additive generalization of the theory of finite non-negative countably additive measures". These $D$-submeasures are not subadditive in general, but we show in the sequel that, except possibly for some less important cases, every $D$-submeasure $\mu$ is equivalent to an order continuous (subadditive!) submeasure $\eta$ such that, in addition, $\mu \ll \eta$.

Dobrakov considers monotone set functions $\mu\colon \mathscr{R} \to \boldsymbol{R}_+$ having the property he calls *subadditive continuity* (sc) or the stronger property of *uniform subadditive continuity* (usc):

(sc) For every $A \in \mathscr{R}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(*)\qquad \mu(A \cup B) \leqslant \mu(A) + \varepsilon \quad \text{and} \quad \mu(A) \leqslant \mu(A \setminus B) + \varepsilon$$

whenever $\mu(B) < \delta$.

(usc) For every $\varepsilon > 0$ there is a $\delta > 0$ such that $(*)$ holds for all $A \in \mathscr{R}$ and all $B \in \mathscr{R}$ with $\mu(B) < \delta$.

If $\mu$ is monotone, order continuous and satisfies (sc) (respectively, (usc)), then it is called a *D-submeasure* (respectively, a *uniform D-submeasure*) on $\mathscr{R}$.

It is easily seen that (sc) and (usc) can be given the following more suggestive forms:

(sc) If $A \in \mathscr{R}$, $(A_n) \subset \mathscr{R}$ and $\mu(A \Delta A_n) \to 0$, then

$$\mu(A_n) \to \mu(A),$$

or

$$\forall A \in \mathscr{R},\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall C \in \mathscr{R}:\ \mu(A \Delta C) < \delta \Rightarrow |\mu(A) - \mu(C)| < \varepsilon.$$

(usc) If $(A_n), (B_n) \subset \mathscr{R}$ and $\mu(A_n \Delta B_n) \to 0$, then

$$\mu(A_n) - \mu(B_n) \to 0,$$

or

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall A, B \in \mathscr{R}:\ \mu(A \Delta B) < \delta \Rightarrow |\mu(A) - \mu(B)| < \varepsilon.$$

Suppose that $\mu: \mathscr{R} \to \boldsymbol{R}_+$ is merely monotone (and $\mu(\emptyset) = 0$ as always). Then we may associate with it a uniformity $\mathfrak{U}(\mu)$ on $\mathscr{R}$, a base of which is formed by the classes

$$\mathscr{U}_\delta = \{(A, B) \in \mathscr{R} \times \mathscr{R}:\ \mu(A \Delta B) < \delta\}, \qquad \delta > 0.$$

Clearly, $\mathfrak{U}(\mu)$ is semimetrizable, and since each $\mathscr{U}_\delta$ is invariant, i.e.,

$$(A, B) \in \mathscr{U}_\delta \Rightarrow (A \Delta C, B \Delta C) \in \mathscr{U}_\delta,$$

there exists a semimetric $d$ on $\mathscr{R}$ generating $\mathfrak{U}(\mu)$, which is also invariant:

$$d(A, B) = d(A \Delta C, B \Delta C) \quad \text{for all } A, B, C \in \mathscr{R}.$$

Let $\Gamma(\mu)$ denote the topology on $\mathscr{R}$ associated with $\mathfrak{U}(\mu)$ (or with $d$). Clearly, $\Gamma(\mu)$ is the weakest invariant topology on $\mathscr{R}$ with respect to which $\mu$ is continuous at $\emptyset$, that is

$$\big(E_n \to \emptyset \text{ in } \Gamma(\mu)\big) \Leftrightarrow \big(\mu(E_n) \to 0\big).$$

Note also that $E_n \to E$ in $\Gamma(\mu)$ iff $\mu(E_n \Delta E) \to 0$.

3.1. PROPOSITION 1. *Let* $\mu: \mathscr{R} \to \boldsymbol{R}_+$ *be monotone. Then*

(i) $\Gamma(\mu)$ *is an* FN-*topology, i.e., there exists a submeasure* $\eta$ *on* $\mathscr{R}$ *such that*

$$\mu(E_n) \to 0 \Leftrightarrow \eta(E_n) \to 0,$$

*iff* $\mu$ *satisfies the following condition*:

(ac) *If* $\mu(A_n) + \mu(B_n) \to 0$, *then* $\mu(A_n \cup B_n) \to 0$.

(ii) $\mu$ *is continuous on* $(\mathscr{R}, \Gamma(\mu))$ *iff* $\mu$ *satisfies* (sc).

(iii) $\mu$ *is uniformly continuous on* $(\mathscr{R}, \Gamma(\mu))$ *iff* $\mu$ *satisfies* (usc).

Proof. (i) The classes

$$\mathcal{U}_\delta(\emptyset) = \{A \in \mathcal{R}:\ \mu(A) < \delta\}$$

form a base of $\Gamma(\mu)$-neighbourhoods of $\emptyset$, and each of them is solid (or normal):

$$B \subset A \in \mathcal{U}_\delta(\emptyset) \Rightarrow B \in \mathcal{U}_\delta(\emptyset).$$

Hence, by [4], 1.5, a necessary and sufficient condition for $\Gamma(\mu)$ to be an FN-topology is that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$A, B \in \mathcal{U}_\delta(\emptyset) \Rightarrow A \cup B \in \mathcal{U}_\varepsilon(\emptyset).$$

This is evidently equivalent to (ac). Since $\Gamma(\mu)$ is semimetrizable, $\Gamma(\mu)$ is an FN-topology iff there is a submeasure $\eta$ for which $\Gamma(\mu) = \Gamma(\eta)$, by [4], 2.3, and such a submeasure can be defined by

$$\eta(E) = \sup\{d(F, \emptyset):\ F \subset E,\ F \in \mathcal{R}\},$$

where $d$ is any invariant semimetric determining $\Gamma(\mu)$ (cf. the proof of 2.3 in [4]).

(ii) and (iii) are obvious.

In view of (i), it is important to know when $\mu$ satisfies condition (ac).

**3.2.** PROPOSITION 2. *A monotone set function $\mu: \mathcal{R} \to \boldsymbol{R}_+$ satisfies* (ac) *in each of the following cases:*

(1) *$\mu$ satisfies* (usc).

(2) *$\mathcal{R}$ is a $\sigma$-ring, $E_n \nearrow E$ implies $\mu(E_n) \to \mu(E)$, and for every $A \in \mathcal{R}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\mu(A \cup B) \leqslant \mu(A) + \varepsilon \quad \textit{whenever } \mu(B) < \delta.$$

(3) *$\mu$ has the property*

(+) *If $(C_n)$ is an increasing sequence in $\mathcal{R}$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\mu(C_n \cup B) \leqslant \mu(C_n) + \varepsilon$$

*for all $n$ and all $B \in \mathcal{R}$ with $\mu(B) < \delta$* (see [3], p. 21).

Proof. (1) is easy, and (2) is a consequence of Theorem 3,b in [3]. For (3) suppose that $\mu(A_n) + \mu(B_n) \to 0$ but $\mu(A_n \cup B_n) \nrightarrow 0$. Then, by passing to subsequences, we may assume that for some $\varepsilon > 0$

$$\mu(A_n \cup B_n) > \varepsilon \quad \text{for all } n \in \boldsymbol{N}$$

and that for every $r \in \boldsymbol{N}$

$$\mu\Big(\bigcup_{n=r}^{k} A_n\Big) < \frac{1}{r} \quad \text{for all } k \geqslant r.$$

(Here we use $(+)$ for constant sequences $(C_n)$.) Choose $m \in \boldsymbol{N}$ so that $(1/m) < \varepsilon/2$, and set

$$D_k = \bigcup_{n=m}^{m+k} A_n \quad \text{for all } k \in \boldsymbol{N}.$$

Clearly, $D_k \nearrow$ and $\mu(D_k) < \varepsilon/2$ for all $k$. By $(+)$ we find $\delta > 0$ such that

$$\mu(B) < \delta \Rightarrow \forall_k \mu(D_k \cup B) \leqslant \mu(D_k) + \frac{\varepsilon}{2} < \varepsilon.$$

Now let $p \in \boldsymbol{N}$ be such that $\mu(B_n) < \delta$ for all $n \geqslant p$. Then

$$\mu(A_{m+k} \cup B_{m+k}) \leqslant \mu(D_k \cup B_{m+k}) < \varepsilon \quad \text{whenever } m+k \geqslant p;$$

a contradiction.

From 3.1 and 3.2 we get immediately

**3.3.** Corollary 1. *The class of uniform D-submeasures on a ring $\mathscr{R}$ (respectively, the class of D-submeasures on a $\sigma$-ring $\mathscr{R}$) coincides with the class of monotone set functions $\mu$: $\mathscr{R} \to \boldsymbol{R}_+$ for which an order continuous submeasure $\eta$ on $\mathscr{R}$ can be found such that $\mu \sim \eta$ and $\mu$ is uniformly continuous on $(\mathscr{R}, \eta)$ (respectively, $\mu \sim \eta$ and $\mu \ll \eta$).*

It is therefore evident that results of certain type already known to hold for submeasures yield immediately the corresponding similar results for $D$-submeasures. In particular, most of the results on "abstract" $D$-submeasures in Section 1 of [3] can be obtained in this way. Thus, for instance, Dobrakov's Theorems 4, 5, 6, 14, 15 follow from [4], 4.8, 6.1, 6.7, 3.1, 7.1, respectively. Actually, only the order continuity of $\mu$ and condition (ac) are essential here; (sc) is either superfluous or can be replaced by the weaker property:

$$\mu(B) = 0 \Rightarrow \mu(A \cup B) = \mu(A) \quad \text{for all } A, B \in \mathscr{R}.$$

It is less evident that also one of the main results of [3], the extension Theorem 18, can be deduced from the extension theorem for submeasures (see [2], and [4], 7.2). We are going to show this, thus giving a "topological" proof of Theorem 18 in [3].

**3.4.** Theorem 1 (Dobrakov). *Let $\mu_0$: $\mathscr{R} \to \boldsymbol{R}_+$ be a D-submeasure and let $\mathscr{S}$ be the $\sigma$-ring generated by $\mathscr{R}$. Then $\mu_0$ can be extended to a D-submeasure $\mu$: $\mathscr{S} \to \boldsymbol{R}_+$ iff the following conditions are fulfilled:*

(i) *$\mu_0$ is exhaustive.*

(ii) *$\mu_0$ has property $(+)$ from Proposition 2.*

(iii) *If $(A_n) \subset \mathscr{R}_\sigma$ and $A_n \searrow$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu^+(A_n \setminus B) \geqslant \mu^+(A_n) - \varepsilon$ for all $n \in \boldsymbol{N}$ and all $B \in \mathscr{R}$ with $\mu_0(B) < \delta$.*

*The extension $\mu$, if it exists, is unique.*

Here $\mathscr{R}_\sigma$ denotes the class of sets which are limits of increasing sequences in $\mathscr{R}$, and $\mu^+ \colon \mathscr{R}_\sigma \to \boldsymbol{R}_+$ is defined by

$$\mu^+(A) = \lim_{n\to\infty} \mu_0(A_n),$$

where $(A_n)$ is any sequence in $\mathscr{R}$ such that $A_n \nearrow A$. Thus $\mu^+$ is well defined, finite valued and extends $\mu_0$ (cf. Lemma A on p. 21-22 in [3]).

Proof. Necessity. Suppose that $\mu$ is a required extension of $\mu_0$ to $\mathscr{S}$. Then, by 3.3, there exists an order continuous submeasure $\eta$ on $\mathscr{S}$ such that $\mu \sim \eta$ and $\mu \ll \eta$. It is obvious that $\mu$ is exhaustive, and hence so is $\mu_0$.

Suppose that condition (ii) is not satisfied. Then, as it is easily seen, there exist an increasing sequence $(A_n)$ in $\mathscr{R}$, an $\varepsilon > 0$ and a sequence $(B_n)$ in $\mathscr{R}$ with $\mu_0(B_n) \to 0$ such that

$$\mu_0(A_n \cup B_n) > \mu_0(A_n) + \varepsilon \quad \text{for all } n \in \boldsymbol{N}.$$

Let

$$A = \bigcup_{n=1}^{\infty} A_n;$$

then

$$\eta\big((A_n \cup B_n) \,\Delta\, A\big) \leqslant \eta(A \setminus A_n) + \eta(B_n) \to 0.$$

Since $\mu \ll \eta$, we get

$$\mu_0(A_n \cup B_n) \to \mu(A) \quad \text{and} \quad \mu_0(A_n) \to \mu(A),$$

but this contradicts our assumption.

Similarly, we may verify that $\mu_0$ must satisfy (iii).

Sufficiency. By Proposition 2 (3) and Proposition 1 (i) and (ii), there exists an order continuous submeasure $\eta_0$ on $\mathscr{R}$ such that $\mu_0 \sim \eta_0$ and $\mu_0 \ll \eta_0$. Let $\eta$ be the unique order continuous extension of $\eta_0$ to $\mathscr{S}$ (see [2], and [4], 7.2).

If there exists a required extension $\mu$ of $\mu_0$, and if $\lambda$ is an order continuous submeasure on $\mathscr{S}$ equivalent to $\mu$, then

$$\lambda(E_n) \to 0 \Leftrightarrow \mu(E_n) \to 0,$$

whence $(\lambda|\mathscr{R}) \sim \mu_0 \sim \eta_0$, and so $\lambda \sim \eta$ (see [4], 7.3). Thus $\mu$, if it exists, is a continuous (hence unique) extension of $\mu_0$ from $(\mathscr{R}, \eta_0)$ to $(\mathscr{S}, \eta)$. Therefore, as $\mathscr{R}$ is dense in $(\mathscr{S}, \eta)$ ([4], 7.1) and $(\mathscr{S}, \eta)$ is complete ([4], 3.1, 5.2), to prove the existence of such a continuous extension $\mu$ it is necessary and sufficient to show that

(C) If $(A_n) \subset \mathscr{R}$ is such that $\mu_0(A_n \,\Delta\, A_m) \to 0$ (which is equivalent to $\eta_0(A_n \,\Delta\, A_m) \to 0$), then $\lim_{n\to\infty} \mu_0(A_n)$ exists in $\boldsymbol{R}_+$.

Of course, there is nothing to prove when $\mu_0$ is a uniform $D$-submeasure.

Suppose that a sequence $(B_n) \subset \mathscr{R}$ is such that

(1) $$\mu_0\Big(\bigcup_{i=n}^{k-1}(B_{i+1}\Delta B_i)\Big) < \frac{1}{n}$$

holds for all $n \in \boldsymbol{N}$ and $k > n$.

Take $\varepsilon > 0$ and then, for each $n \in \boldsymbol{N}$, choose $\delta_n > 0$ so that

$$\forall C \in \mathscr{R}\colon\ \mu_0(C) < \delta_n \Rightarrow \mu_0(B_n \cup C) \leqslant \mu_0(B_n) + \varepsilon;$$

this is possible by (sc). Then, if $C \in \mathscr{R}_\sigma$ and $\mu^+(C) < \delta_n$, we also have

$$\mu^+(B_n \cup C) \leqslant \mu_0(B_n) + \varepsilon.$$

For each $n \in \boldsymbol{N}$ and $k \geqslant n$ let

$$C_{nk} = \bigcup_{i=n}^{k} B_i \quad \text{and} \quad C_n = \bigcup_{i=n}^{\infty} B_i.$$

From the exhaustivity of $\mu_0$ it follows easily that

$$\mu^+(C_n \setminus C_{nk}) \to 0 \quad \text{as } k \to \infty.$$

Hence for each $n \in \boldsymbol{N}$ there exists $k_n \geqslant n$ such that

$$\mu^+(C_n \setminus C_{nk}) < \delta_n \quad \text{for all } k \geqslant k_n.$$

We have

(2) $$\mu^+\big(B_n \cup (C_n \setminus C_{nk})\big) \leqslant \mu_0(B_n) + \varepsilon \quad \text{for all } n \in \boldsymbol{N} \text{ and } k \geqslant k_n.$$

By condition (iii) there exists $\delta > 0$ such that

$$\mu^+(C_n \setminus B) \geqslant \mu^+(C_n) - \varepsilon$$

for all $n$ and all $B \in \mathscr{R}$ with $\mu_0(B) < \delta$. Now, since

$$C_{nk} \setminus B_n \subset \bigcup_{i=n}^{k-1}(B_{i+1}\Delta B_i),$$

we have $\mu_0(C_{nk} \setminus B_n) < n^{-1}$ for all $n$ and $k \geqslant n$. Choose $m \in \boldsymbol{N}$ with $m^{-1} < \delta$; then, if $n \geqslant m$ and $k \geqslant n$, we have

(3) $$\mu^+\big(C_n \setminus (C_{nk} \setminus B_n)\big) \geqslant \mu^+(C_n) - \varepsilon.$$

Since

$$C_n \setminus (C_{nk} \setminus B_n) = B_n \cup (C_n \setminus C_{nk}),$$

combining (2) and (3) (for $k = k_n$) we get

$$0 \leqslant \mu^+(C_n) - \mu_0(B_n) \leqslant 2\varepsilon \quad \text{for all } n \geqslant m.$$

Since $C_n \searrow$, $\lim \mu^+(C_n)$ exists, and so does $\lim \mu_0(B_n)$.

Now let $(A_n)$ be as in (C), and let $(A_n')$ and $(A_n'')$ be two arbitrary subsequences of $(A_n)$. Then a simple argument using $\sigma$-subadditivity of $\eta$ (cf. also the proof of Theorem 14 in [3]) shows that we may select a subsequence $(B_n)$ of $(A_n)$ which satisfies (1) and contains infinitely many terms of each of $(A_n')$ and $(A_n'')$. By the preceding part of the proof, $g = \lim \mu_0(B_n)$ exists, and our argument implies easily that every subsequence of $(\mu_0(A_n))$ has a subsequence converging to $g$. Hence $\lim \mu_0(A_n)$ exists and equals $g$.

The required extension $\mu$ of $\mu_0$ is thus defined as follows: Given $A \in \mathscr{S}$, choose $(A_n)$ in $\mathscr{R}$ such that $\eta(A \,\Delta\, A_n) \to 0$ and then put

$$\mu(A) = \lim \mu_0(A_n).$$

Since $\mu$ extends $\mu_0$, and $\mu$ is $\eta$-continuous, $\mu$ is evidently order continuous. So it remains only to verify that $\mu$ is monotone. But this is an easy consequence of monotonicity of $\mu^+$ and of the fact that for every $A \in \mathscr{S}$ there exists $(A_n) \subset \mathscr{R}_\sigma$ with $A \subset A_n$ for each $n$, $A_n \searrow$ and $\eta(A_n \setminus A) \to 0$.

Remark. As the proof shows, condition (ii) can be replaced by (ac). It is not clear to the author whether conditions (ii) and (iii) are independent of the other properties of $\mu_0$.

4. In this section we give an affirmative answer to the problem posed by Dobrakov on p. 14 in [3], i.e., we prove the following

THEOREM 2. *Let $\mathscr{S}$ be a $\sigma$-ring of subsets of a set $T$ and let $\mu$ be a D-submeasure on $\mathscr{S}$. Then the range of $\mu$, $\mu(\mathscr{S}) = \{\mu(A): A \in \mathscr{S}\}$ is a compact subset of $\boldsymbol{R}_+$.*

Since $\mu$ is bounded ([3], Theorem 4), we have only to show that $\mu(\mathscr{S})$ is closed.

LEMMA 1. *The range of every D-submeasure $\lambda$ on the $\sigma$-algebra $\mathscr{P}(\boldsymbol{N})$ of all subsets of $\boldsymbol{N}$ is compact.*

Proof. By 3.1 and 3.2, there exists an order continuous submeasure $\eta$ on $\mathscr{P}(\boldsymbol{N})$ such that $\lambda \ll \eta$. Now, in turn, $\eta$ is continuous with respect to the $\sigma$-additive measure $\nu$ defined by

$$\nu(A) = \sum_{n \in A} 2^{-n},$$

and hence $\lambda$ is also $\nu$-continuous. It is well known that $(\mathscr{P}(\boldsymbol{N}), \nu)$ is compact and, therefore, so is the range of $\lambda$.

LEMMA 2 (cf. [3], Theorem 10). *Let $\mathscr{S}$ and $\mu$ be as in Theorem 2. Suppose that $P, Q \in \mathscr{S}$, $Q$ contains no atoms of $\mu$ and $\mu(P) < \mu(P \cup Q)$. Then for each $a$,*

$$\mu(P) < a < \mu(P \cup Q),$$

*there exists $B$ in $\mathscr{S}$ such that $B \subset Q$ and $\mu(P \cup B) = a$.*

Proof. Recall that a set $A \in \mathscr{S}$ is said to be an *atom* of $\mu$ if $\mu(A) > 0$ and either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$ for every $B \in \mathscr{S}$, $B \subset A$. We may assume that $P \cap Q = \emptyset$. Let $\Delta$ be the class of all families $\mathscr{D} \subset \mathscr{S}$ which consist of pairwise disjoint subsets $D$ of $Q$ with $\mu(D) > 0$ and are such that $\mu(P \cup \bigcup \mathscr{D}) \leqslant a$. Clearly, each $\mathscr{D} \in \Delta$ is at most countable. Let $\Delta_0$ be a chain in $(\Delta, \subset)$. Then $\mathscr{D}_0 = \bigcup \Delta_0$ is at most countable, and hence there exists an increasing sequence $(\mathscr{D}_n)$ in $\Delta_0$ such that

$$\bigcup_{n=1}^{\infty} \mathscr{D}_n = \mathscr{D}_0.$$

Clearly,

$$\mu\left(P \cup \bigcup \mathscr{D}_0\right) = \lim_{n\to\infty} \mu\left(P \cup \bigcup \mathscr{D}_n\right) \leqslant a,$$

so that $\mathscr{D}_0 \in \Delta$ and $\mathscr{D} \subset \mathscr{D}_0$ for all $\mathscr{D} \in \Delta_0$.

By the Kuratowski-Zorn Principle there exists a maximal element $\mathscr{D}_0$ in $\Delta$. Let $B = \bigcup \mathscr{D}_0$; we have $\mu(P \cup B) \leqslant a$. Suppose that $\mu(P \cup B) < a$. Then, since $\mu$ is atomless on $Q \setminus B$ and $\mu(Q \setminus B) > 0$, there exists in $\mathscr{S}$ a subset $C$ of $Q \setminus B$ such that

$$\mu(C) > 0 \quad \text{and} \quad \mu(P \cup B \cup C) < a.$$

(We use the Saks decomposition ([3], Theorem 8) and (sc).) It follows that $\mathscr{D}_0$ is not maximal in $\Delta$; a contradiction.

Proof of Theorem 2. By Theorem 4 in [3] we may suppose that $T \in \mathscr{S}$. Let $T = T_a \cup T_b$ be a decomposition of $T$ into a purely atomic part $T_a$ and an atomless part $T_b$. Thus $T_a$, if it is non-empty, is the union of at most countably many pairwise disjoint atoms of $\mu$.

Case 1. $T_a = \emptyset$. Then $\mu(\mathscr{S}) = [0, \mu(T)]$ by Lemma 2 ($P = \emptyset$, $Q = T$) or by Theorem 10 in [3].

Case 2. $T_b = \emptyset$ (or $\mu(T_b) = 0$). Then $\mu(\mathscr{S})$ is closed by Lemma 1.

Case 3. $T_a \neq \emptyset \neq T_b$. Let $r_n = \mu(A_n \cup B_n) \to r$, where $A_n \subset T_a$ and $B_n \subset T_b$. We claim that $r \in \mu(\mathscr{S})$. This is non-trivial if $r > \mu(T_b)$ and $r_n \neq r$ for all $n$. Assuming this, we shall consider two subcases, each with two sub-subcases.

(a) A subsequence of $(r_n)$, and we may assume that also the sequence $(r_n)$ itself, satisfies $r_n > r$ for all $n$.

(a′) There exists $m \in N$ with $\mu(A_m) < r$. Then, according to Lemma 2, we can find $B \subset B_m$ such that $\mu(A_m \cup B) = r$. Thus $r \in \mu(\mathscr{S})$.

(a″) $\mu(A_n) \geqslant r$ for all $n$. Then $\mu(A_n) \to r$ and $r \in \mu(\mathscr{S})$ since $\mu(T_a \cap \mathscr{S})$ is closed by Lemma 1.

(b) For almost all $n$, and we may assume that also for all $n$, we have $r_n < r$.

(b′) There exists $m \in N$ with $\mu(A_m \cup T_b) \geqslant r$. Then Lemma 2 gives us a subset $B$ of $T_b$ with $\mu(A_m \cup B) = r$, and thus $r \in \mu(\mathscr{S})$.

(b″) $\mu(A_n \cup T_b) < r$ for all $n$. Then $\mu(A_n \cup T_b) \to r$. Treating $T_b$ as a "point" (the details are clear), we apply Lemma 1 and see that $r \in \mu(\mathscr{S})$.

**5.** Let $X = (X, |\cdot|)$ be a quasi-normed Abelian group. A set function $\mu: \mathscr{R} \to X$ (with $\mu(\emptyset) = 0$) is said to be *quasi-Lipschitzian* with constant $N \in \boldsymbol{R}_+$ if

(qL) $\quad |\mu(A \cup B) - \mu(A)| \leqslant N|\mu(B)| \quad$ for all disjoint $A, B \in \mathscr{R}$,

and it is said to be *N-triangular* with $N \in \boldsymbol{R}_+$ if

(Nt) $\quad |\mu(A)| - N|\mu(B)| \leqslant |\mu(A \cup B)| \leqslant |\mu(A)| + N|\mu(B)|$

for all disjoint $A, B \in \mathscr{R}$.

(Cf. [1], [5] and the references given there.)

Evidently, (qL) $\Rightarrow$ (Nt). Moreover, if $\mu$ is $N$-triangular, then the set function $|\mu(\cdot)|: A \mapsto |\mu(A)|$ is quasi-Lipschitzian with constant $N$.

Let $\mu: \mathscr{R} \to X$ be quasi-Lipschitzian; then the set function $\bar{\mu}: \mathscr{R} \to \overline{\boldsymbol{R}}_+$ defined by

$$\bar{\mu}(E) = \sup\{|\mu(F)|: F \subset E, F \in \mathscr{R}\}$$

is monotone, $\bar{\mu}(\emptyset) = 0$, and satisfies

$$\bar{\mu}(A \cup B) \leqslant \bar{\mu}(A) + N\bar{\mu}(B) \quad \text{for all } A, B \in \mathscr{R}$$

(hence also (Nt)). It follows from Proposition 1 (i) (which is obviously valid also if $\mu$ is $\overline{\boldsymbol{R}}_+$-valued) that there exists a submeasure $\eta$ on $\mathscr{R}$ such that $\bar{\mu} \sim \eta$. Since (qL) implies that

$$|\mu(E) - \mu(F)| \leqslant N\big(|\mu(E \setminus F)| + |\mu(F \setminus E)|\big) \quad \text{for all } E, F \in \mathscr{R},$$

and hence

$$|\mu(E) - \mu(F)| \leqslant 2N\bar{\mu}(E \,\Delta\, F) \quad \text{for all } E, F \in \mathscr{R},$$

the function $\mu: (\mathscr{R}, \eta) \to (X, |\cdot|)$ is uniformly continuous. If $\mu$ is exhaustive or order continuous, then so is $\bar{\mu}$ (cf. [5]), and hence so is $\eta$.

We give, as examples of possible applications, two results on quasi-Lipschitzian and $N$-triangular set functions. The first one improves a theorem stated in Section 2 of [5]; its proof uses the method of extension by continuity and is clear enough so that we do not include it here. The other one is a Nikodym type theorem for $N$-triangular set functions (cf. [1], p. 671); we formulate it for $\boldsymbol{R}_+$-valued functions only, since this is essentially the most important case.

THEOREM 3. *Every order continuous, exhaustive and quasi-Lipschitzian set function $\mu_0: \mathscr{R} \to X$, where $X$ is a complete normed Abelian group, has a unique order continuous quasi-Lipschitzian* (*with the same constant $N$*) *extension $\mu$ to the $\sigma$-ring $\mathscr{S}$ generated by $\mathscr{R}$.*

THEOREM 4. *Let $M$ be a family of order continuous $N$-triangular set functions $\mu\colon \mathscr{S} \to \mathbf{R}_+$, where $\mathscr{S}$ is a $\sigma$-ring, such that*

$$\sup\{\mu(A)\colon \mu \in M\} < \infty \quad \textit{for every } A \in \mathscr{S}.$$

*Then*

$$\sup\{\mu(A)\colon \mu \in M,\ A \in \mathscr{S}\} < \infty.$$

Proof (sketch). We may easily reduce the proof to the case where $M$ is countable. Then there exists an order continuous submeasure $\eta$ on $\mathscr{S}$ such that each $\mu \in M$ is $\eta$-continuous. Since $(\mathscr{S}, \eta)$ is complete, by a standard Baire category argument we find that the functions in $M$ are uniformly bounded in a $\Gamma(\eta)$-neighbourhood of a set $A_0 \in \mathscr{S}$. Then, using (Nt), we see that they are also uniformly bounded in a neighbourhood of $\emptyset$. Applying the Saks decomposition for $\eta$ ([3], Theorem 8) and (Nt), we quickly arrive at the conclusion of the theorem.

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