COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

A CLASSIFICATION OF MEASURE SPACES

BY

ROBERT E. ZINK* (LAFAYETTE, IND.)

1. Introduction. According to the classical theorem of Vitali and Carathéodory, every bounded Lebesgue measurable function on $[0,1]$ is measure-theoretically equivalent to a function of Baire type 2. In a Boolean space with an appropriate Borel measure, it is known that each bounded measurable function is almost everywhere equal to a continuous function. Motivated by these theorems, one might term a topological measure space an *a-space* if each bounded measurable function defined thereon were equivalent to a function of Baire type a and if a were the smallest ordinal for which this were true. The aforementioned spaces would then furnish examples of a-spaces for $a = 0$ and $a = 2$. An interesting question springs immediately to the mind: Do there exist a-spaces for other ordinals a? This seems to be a difficult problem.

One proof of the Vitali-Carathéodory theorem shows that each bounded measurable function on $[0,1]$ is equivalent to the limit of a monotone sequence of semicontinuous functions. This suggests a classification of functions and measure spaces that seems to be more natural for the consideration of this sort of problem. The continuous functions, the semicontinuous functions and the limits of sequences of semicontinuous functions are of types 0, 1 and 2. The functions of higher types are obtained by taking limits of sequences of functions of lower types. An a-space is then defined just as before except that one uses as approximating functions the functions of type a according to this new scheme. The spaces mentioned above again supply examples of a-spaces for $a = 0$ and $a = 2$ in this new classification. Moreover, it is found that the unit interval provided with the density topology and Lebesgue measure is a 1-space.

Another proof of the Vitali-Carathéodory theorem depends on Lusin's theorem. This suggests that the possibility of approximation of measurable functions by functions of various types a is determined by

* While this article was being prepared, the author was supported by a grant from the National Science Foundation.

certain regularity properties of the measure. This does turn out to be the case as the theorems of sections 3 and 4 show. In particular, it is found that in the case of an outer regular measure, the associated topological measure space is necessarily of type 2 or less.

2. Preliminary considerations. Definition 1. A topological measure space is a topological space $(X, \boldsymbol{T})$ and a measure space (X, Σ, μ) such that Σ is a σ-algebra that contains $\boldsymbol{T}$, and μ assigns positive measure to each non-empty element of $\boldsymbol{T}$. The quadruple $(X, \boldsymbol{T}, \Sigma, \mu)$ is used to denote the topological measure space.

In all that follows, only finite measures will be considered.

Definition 2. Let $(X, \boldsymbol{T})$ be a topological space. For each non-negative ordinal α, the classes $\mathscr{L}_\alpha(\mathscr{U}_\alpha)$ of real-valued functions on X are defined inductively as follows: $\mathscr{L}_1(\mathscr{U}_1)$ is the class of all lower- (upper-) semicontinuous functions; if α exceeds 1 and if the classes $\mathscr{L}_\beta(\mathscr{U}_\beta)$ have been defined for all $\beta < \alpha$, then $\mathscr{L}_\alpha(\mathscr{U}_\alpha)$ is the class of all limits of convergent sequences of elements of $\bigcup_{\beta<\alpha}\mathscr{L}_\beta(\bigcup_{\beta<\alpha}\mathscr{U}_\beta)$. A member of $\mathscr{L}_\alpha \cup \mathscr{U}_\alpha$ is said to be of type α.

If the class of all continuous real-valued functions defined on X be denoted by $\mathscr{L}_0$ $(= \mathscr{U}_0)$, then it is clear that $\mathscr{L}_0 \subset \mathscr{L}_1 \subset \ldots \subset \mathscr{L}_\Omega$ and that $\mathscr{U}_0 \subset \mathscr{U}_1 \subset \ldots \subset \mathscr{U}_\Omega$, where Ω is the first uncountable ordinal. Since every function of Baire type 1 is simultaneously an element of $\mathscr{L}_2$ and $\mathscr{U}_2$, it follows that every Baire function can be found among the classes $\mathscr{L}_\alpha \cap \mathscr{U}_\alpha$. However, it will be noted in the sequel that there are semicontinuous functions on some topological spaces that are not Baire functions of any order. Thus, these classification schemes are distinct from the Baire scheme. Two measurable sets are said to be *equivalent* $(E \sim F)$ if their symmetric difference has measure zero. Two measurable functions are *equivalent* $(f \sim g)$ if they are almost everywhere equal. The equivalence sets engendered by a measurable set E and a measurable function f are denoted by $[E]$ and $[f]$.

Definition 3. A topological measure space $(X, \boldsymbol{T}, \Sigma, \mu)$ has the property P_β $(0 \leqslant \beta < \Omega)$ if each bounded real-valued function defined on X and measurable (Σ) is equivalent to an element of $\mathscr{L}_\beta$. If $\alpha = \inf\{\beta\colon (X, \boldsymbol{T}, \Sigma, \mu)$ has the property $P_\beta\}$, then $(X, \boldsymbol{T}, \Sigma, \mu)$ is termed an *α-space*.

REMARK 4. *If $(X, \boldsymbol{T}, \Sigma, \mu)$ has the property P_β, then every bounded real-valued function defined on X and measurable (Σ) is equivalent to an element of U_β.*

Proof. On first noting that the negative of a lower-semicontinuous function is upper-semicontinuous, it is easily established by transfinite induction that the negative of a member of $\mathscr{L}_\alpha$ is an element of $\mathscr{U}_\alpha$, for all $\alpha < \Omega$. If $(X, \boldsymbol{T}, \Sigma, \mu)$ has property P_β and if f is a bounded measurable (Σ) function, then $-f$ is equivalent to an element of $\mathscr{L}_\beta$, whence f is almost everywhere equal to a member of $\mathscr{U}_\beta$.

3. The outer regular case. If the measure μ associated with a topological measure space $(X, \boldsymbol{T}, \Sigma, \mu)$ is outer regular, that is to say, if

$$\mu(E) = \inf\{\mu(U): E \subset U,\ U \in \boldsymbol{T}\},$$

for all E in Σ, then the bounded measurable (Σ) functions have a particularly simple structure. Indeed, the following theorem is obtained as a corollary of the other theorems of this section:

THEOREM 5. *If μ is outer regular, then the topological measure space $(X, \boldsymbol{T}, \Sigma, \mu)$ is an a-space with $a \leqslant 2$.*

Moreover, the several examples of this section demonstrate the existence of a-spaces for $a = 0$, $a = 1$ and $a = 2$.

THEOREM 6. *In order that the topological measure space $(X, \boldsymbol{T}, \Sigma, \mu)$ be a 0-space, it is both necessary and sufficient that each measurable subset of X be equivalent to a set that is both closed and open.*

PRELIMINARY REMARK. Under the general hypotheses of Theorem 6, the requirement that each measurable set be equivalent to a closed-and-open set is sufficient to guarantee that the closure of each open set is again open. (In the language of point set topology, the topological space $(X, \boldsymbol{T})$ is extremally disconnected.)

Proof of remark. Let V be an open set. By hypothesis there exists a closed-and-open set U such that $U \sim \mathrm{cl}\, V$. Since $U - \mathrm{cl}\, V$ is an open null set, it follows that U is a subset of $\mathrm{cl}\, V$. Because $\mathrm{cl}\, V - U$ is a null set and because U is closed, $V - U$ is also an open set of zero measure; thus, V is a subset of U. Finally, the relations of inclusion $V \subset U \subset \mathrm{cl}\, V$ imply that $U = \mathrm{cl}\, V$.

Proof of sufficiency. If $f = \chi_E$, with E in Σ, let $g = \chi_U$, where U is a closed-and-open set that is equivalent to E. Clearly, g is continuous, and g is equivalent to f. It follows at once that the equivalence set determined by a simple function always contains a continuous element.

If f is a non-negative, bounded measurable (Σ) function, then, according to one of the most ancient theorems of measure theory, there exists a non-decreasing sequence of simple functions, $\{f_n\}$, converging to f. For each natural number n, let g_n be a continuous element of $[f_n]$. Since

$$\big\{x: g_n(x) > \sup\{f(y): y \in X\}\big\} \subset \{x: g_n(x) > f(x)\} \subset \{x: g_n(x) > f_n(x)\},$$

and since the first of these sets is open while the last has measure zero, the functions g_n are uniformly bounded above. Now, according to a theorem of Stone [14], if $(X, \boldsymbol{T})$ is an extremally disconnected topological space and if $(\mathscr{L}_0; \leqslant)$ is the lattice of continuous real-valued functions associated with $(X, \boldsymbol{T})$, then a non-void subset of $\mathscr{L}_0$ that has an upper bound in

$(\mathscr{L}_0; \leqslant)$ has also a least upper bound there. Thus, $\{g_n\}$ has a least upper bound, g, in $(\mathscr{L}_0; \leqslant)$. From the method of choice of the g_n it follows that $g(x) \geqslant f(x)$ for almost all x in X. Let ε be a positive real number, and let

$$E = \{x\colon g(x) \geqslant f(x) + \varepsilon\}.$$

For each natural number k, let

$$F_k = \{x\colon g(x) \geqslant g_k(x) + \varepsilon\}.$$

Since $g_k(x) \leqslant f(x)$ a.e., it follows that $\mu(F_k) \geqslant \mu(E)$, for all k. If $F = \bigcap_{k=1}^{\infty} F_k$, then the preceding observation together with the relation of each g_k to the corresponding f_k imply that $\mu(F) \geqslant \mu(E)$. Let U be a closed-and-open set that is equivalent to F. Because F is closed, the null set $U - F$ is open. But the only open null set is the empty set; thus, U is a subset of F. Now the continuous function

$$h = g - \varepsilon\chi_U$$

is an upper bound of $\{g_n\}$ in $\{\mathscr{L}_0; \leqslant\}$, whence, for every x in X, $h(x) \geqslant g(x)$. Hence, U is empty, F is a null set, and, as a result, E is also null. Since ε was an arbitrarily chosen positive number, it must be the case that

$$\mu\big(\{x\colon g(x) > f(x)\}\big) = 0,$$

and, as a consequence, g is equivalent to f.

Finally, one treats the general case by resolving the measurable function into its positive and negative parts.

Proof of necessity. Let E be an arbitrary element of Σ, and let $f = \chi_E$. If g is a continuous element of $[f]$, then the open set

$$V = \big\{x\colon g(x) \notin \{0, 1\}\big\}$$

has measure zero and is therefore empty. Thus, g is the characteristic function of some measurable set U. Since g is continuous, U is both open and closed. Finally, the equivalence of U and E follows from the equivalence of f and g.

In order to show that the preceding discussion has not been conducted *in vacuo*, the following well-known example is pertinent.

Example 7. Let $(I, \boldsymbol{L}, m)$ be the Lebesgue measure space associated with the unit interval, and let $\big(\boldsymbol{L}(m), m\big)$ be the corresponding measure algebra. The Boolean algebra $\boldsymbol{L}(m)$ is isomorphic to the Boolean algebra $\boldsymbol{A}$ of all closed-and-open subsets of a certain Boolean space X. If T is such an isomorphism, then it follows that the set function μ defined on $\boldsymbol{A}$ by the relation

$$\mu(U) = m(T^{-1}U),$$

for all U in $\boldsymbol{A}$, is a measure on $\boldsymbol{A}$. If Σ is the σ-algebra generated by $\boldsymbol{A}$, then it is known that each element of Σ is uniquely representable as the symmetric difference of an element of $\boldsymbol{A}$ and a first category element of Σ. Hence, μ can be extended to a set function $\tilde{\mu}$ meaningfully defined on all of Σ in the following manner: if E is an element of Σ, then

$$\tilde{\mu}(E) = \mu(U),$$

where $E = U \Delta M$, U is an element of $\boldsymbol{A}$ and M is a set of the first category. It is a not too difficult task to verify that $\tilde{\mu}$ is a measure on Σ that vanishes on the class of all measurable sets of the first category.

Now let $\boldsymbol{T}$ be the topology generated by $\boldsymbol{A}$, and let U be an arbitrary element of $\boldsymbol{T}$. There exists a subset $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $U = \bigcup \{G; G \epsilon \boldsymbol{B}\}$. Since $\boldsymbol{A}$ is a complete (with respect to the Boolean order relation) lattice, $\boldsymbol{B}$ has a least upper bound V, and there follows from the finiteness of the measure, the existence of a denumerable subset $\{V_n: n = 1, 2, \ldots\}$ of $\boldsymbol{A}$ such that $V = \bigvee_{n=1}^{\infty} V_n$, and $W = \bigcup_{n=1}^{\infty} V_n$ is a (measurable) subset of U. It then follows from the definition of μ that $\tilde{\mu}(W) = \tilde{\mu}(V)$; hence, $V - U$ is a subset of the null set $V - W$. Moreover, U is a subset of V so that U is an element of Σ^*, the completion of Σ with respect to $\tilde{\mu}$. Thus, if μ^* is the completion of $\tilde{\mu}$, then $(X, \boldsymbol{T}, \Sigma^*, \mu^*)$ is a 0-space.

In this connection, it should be noted that Mibu [10] has given a proof that the condition of Theorem 6 is sufficient to guarantee that a Boolean space with a suitable measure is a 0-space. (A part of the proof is attributed by Mibu to Ogasawara.)

If the measure μ of a 0-space $(X, \boldsymbol{T}, \Sigma, \mu)$ is complete, then the preceding discussion shows that it is a category measure as defined by Oxtoby [12]. This is a consequence of Oxtoby's remark that the completion of a finite Borel measure m is a category measure if and only if $m(G) = m(\mathrm{cl}\, G)$ for every open set G, and $m(G)$ is positive for every open set of the second category.

THEOREM 8. *Let $(X, \boldsymbol{T}, \Sigma, \mu)$ be a topological measure space. Each bounded real-valued function defined on X and measurable (Σ) is equivalent to a lower-semicontinuous function if and only if each measurable set is equivalent to an open set.*

PRELIMINARY COMMENT. Since the negative of a lower-semicontinuous function is upper-semicontinuous and since the complement of an open set is closed, it is clear that a true proposition can be obtained from Theorem 8 by replacing in the statement of that theorem the word lower by the word upper or the word open by the word closed.

Proof of sufficiency. If $f = \chi_E$, with E an element of Σ, let $g = \chi_U$, where U is open and equivalent to E. Then g is lower-semicontinuous and is almost everywhere equal to f. Since the class of lower-semicontinuous functions is closed under addition and non-negative scalar multiplication, it is an immediate consequence of the foregoing that each simple function is equivalent to a lower-semicontinuous one.

If f is bounded, non-negative and measurable (Σ), let $\{f_n\}$ be a non-decreasing sequence of simple functions having f as limit function. For each natural number n, let g_n be a lower-semicontinuous function that is equivalent to f_n, and let $g = \bigvee_{n=1}^{\infty} g_n$ (the function obtained by taking the pointwise supremum of the functions g_n). Then g is also lower-semicontinuous (see, for example, [8], p. 101), and g is an element of $[f]$. That g is real-valued (and, in fact, bounded) follows from the boundedness of f by means of an argument identical to that used at the corresponding step in the proof of Theorem 6.

In the general case, first note that the route followed above can be retraced using closed sets in place of open ones to show that the equivalence class of each bounded, non-negative measurable (Σ) function contains a bounded upper-semicontinuous element. Thus, if g and h are bounded equivalents of f^+ and f^-, if g is lower-semicontinuous and if h is upper-semicontinuous, then $g-h$ is a lower-semicontinuous equivalent of f.

Proof of necessity. Let E be an arbitrary element of Σ, and let $f = \chi_E$. If g is a lower-semicontinuous element of $[f]$, then the open set $\{x: g(x) > 0\}$ is an element of $[E]$.

Theorem 8 can now be applied to show that the following is an example of a 1-space.

Example 9. Let $(I, \boldsymbol{L}, m)$ be the Lebesgue measure space associated with the unit interval, and let $\boldsymbol{T}^*$ be the density topology on I introduced by Haupt and Pauc [7]. A set U is open in this topology if it is measurable and if the metric density of U exists and is equal to 1 at each of its points. Thus, it follows from the Lebesgue Density Theorem and Theorem 8, that $(I, \boldsymbol{T}^*, \boldsymbol{L}, m)$ is an α-space with $\alpha \leqslant 1$. This topological measure space is not a 0-space, because I is connected with respect to the topology $\boldsymbol{T}^*$ [6], so that there are no non-trivial closed-and-open sets.

It is interesting to note that Blumberg has observed in [1] that each Lebesgue measurable function f defined on I is equivalent to an approximately lower-semicontinuous function and to an approximately upper-semicontinuous function. These two functions are the upper and lower measurable boundaries of f constructed by Blumberg. Now the approximately semicontinuous functions are precisely those functions that are semicontinuous with respect to $\boldsymbol{T}^*$; thus, the Blumberg measurable boundaries of a bounded measurable function f are concrete examples of the T^*-semicontinuous equivalents of f, the existence of which is guaranteed by Theorem 8.

One might question whether (I, T^*, L, m) be a 1-space according to the original classification proposed in the introduction. This would certainly be true if every function semicontinuous with respect to the density topology were the limit of a sequence of approximately continuous functions, i.e., a function of Baire class 1 (with respect to T^*). However, the approximately continuous functions are all elements of the first Baire class associated with T, the ordinary topology on I [2]. Thus, if every T^*-semicontinuous function were the limit of a sequence of T^*-continuous functions, then each such function would be an element of the second Baire class associated with T. But if E is a Lebesgue null set that is not a Borel set, then χ_E is T^*-upper-semicontinuous but is not an element of any Baire class (T). Nevertheless, it might be the case that each T^*-semicontinuous function is equivalent to an element of the first Baire class (T^*). By virtue of Theorem 8 and Example 9, this proposition is equivalent to the assertion that (I, T^*, L, m) is a 1-space according to the Baire scheme. Although it will not be considered further here, the question has been answered in the affirmative. A complete discussion of the problem is contained in a forthcoming article [16].

In connection with the above remarks, one might note that the work of Hing Tong [15] can be applied to yield a proof, different from that given by Goffman, Neugebauer, and Nishiura [5], of the fact that the density topology is not normal.

LEMMA 10. *Each T^*-closed set is a T^*-G_δ.*

Proof. Let F be closed in the density topology. Then F is measurable, so that, as a consequence of the outer regularity of the Lebesgue measure, there exists a sequence $\{U_n\}$ of sets open in the Euclidean topology such that each U_n contains F, and $m(\bigcap_{n=1}^{\infty} U_n) = m(F)$. Since $K = \bigcap_{n=1}^{\infty} U_n - F$ is a null set, it is T^*-closed; hence, each of the sets $V_n = U_n - K$ is T^*-open. Finally, $F = \bigcap_{n=1}^{\infty} V_n$.

Now, if T^* were a normal topology, then it would follow from the lemma that it would be perfectly normal. Thus, every lower-semicontinuous function would be the limit of an increasing sequence of continuous functions [15]. Since this is impossible, T^* cannot be normal.

If one is willing to consider atomic measures, then the question of the existence of 1-spaces according to the Baire classification can be easily answered in the affirmative.

Example 11. Let (X, T) be the one-point compactification of the topological space obtained by imposing the discrete topology on N, the set of all natural numbers, and let Σ be the class of all subsets of X. A measure μ is defined on Σ as follows: if E is any subset of X, then, $\mu(E) = \sum_{n \in E \cap N} 2^{-n} \cap N^{2^{-n}}$. It is a simple matter to verify that (X, T, Σ, μ) is a 1-space.

One further question related to Example 9 is worthy of consideration. One notes that T^* is a strengthening of T. Is it possible to find a yet stronger topology T^{**} such that (I, T^{**}, L, m) is a 0-space? This problem remains unsolved (**P 561**).

THEOREM 12. *Let (X, T, Σ, μ) be a topological measure space. In order that each bounded, real-valued measurable (Σ) function be equivalent to an element of $\mathscr{L}_2$, it is both necessary and sufficient that each measurable set be equivalent to a set of type G_δ.*

PRELIMINARY COMMENT. A true proposition can be obtained from the statement of Theorem 12 by replacing $\mathscr{L}_2$ by $\mathscr{U}_2$ or by replacing G_δ by F_σ.

Proof of sufficiency. If E is a measurable set, then E is equivalent to a set G of type G_δ. If $G = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open, then $\chi_E \sim \bigwedge_{n=1}^{\infty} f_n$, where $f_n = \chi_{U_n}$. Since each f_n is lower-semicontinuous, χ_E is equivalent to an element of $\mathscr{L}_2$. Since $\mathscr{L}_2$ is both positively homogeneous and additive, it follows at once that each simple function is equivalent to a member of $\mathscr{L}_2$. In similar fashion, it is easy to see that the class of all functions that are equivalent to elements of $\mathscr{U}_2$ contains the class of all simple functions. If f is an arbitrary bounded non-negative measurable function, then there exist monotone sequences of simple functions $\{f_n\}$, $\{g_n\}$, such that

$$f = \bigvee_{n=1}^{\infty} f_n = \bigwedge_{n=1}^{\infty} g_n.$$

For each n, let h_n be an element of $\mathscr{U}_2$, that is equivalent to f_n, and let k_n be an equivalent of g_n that lies in $\mathscr{L}_2$. Then, $h = \bigvee_{n=1}^{\infty} h_n$ is an element of $\mathscr{U}_2$, $k = \bigwedge_{n=1}^{\infty} k_n$ is an element of $\mathscr{L}_2$ and both h and k are equivalent to f. Because f is bounded and because non-empty open sets have positive measure, h and k are necessarily real-valued.

In the general case, decompose f into its positive and negative parts. From the above, there exist a real-valued element g of $\mathscr{L}_2$ and a real-valued element h of $\mathscr{U}_2$ such that $f \sim g - h$. Since the negative of an element of $\mathscr{U}_2$ is an element of $\mathscr{L}_2$, the proof is complete.

Proof of necessity. Let E be a measurable set, and let f be an element of $\mathscr{L}_2$ that is equivalent to χ_E. Then $f = \lim f_n$ for some sequence $\{f_n\}$ of lower-semicontinuous functions. Since $f \overset{n}{=} \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} f_k$, and, since each of the functions $\bigvee_{k=n}^{\infty} f_k$ is lower-semicontinuous, one may suppose that $\{f_n\}$ is a non-increasing sequence. Now, in this case,

$$\begin{aligned} E &\sim \{x: f(x) = 1\} \sim \{x: f(x) \geqslant \tfrac{1}{4}\} \\ &= \bigcap_{n=1}^{\infty} \{x: f_n(x) \geqslant \tfrac{1}{4}\} \supset \bigcap_{n=1}^{\infty} \{x: f_n(x) > \tfrac{1}{4}\} \\ &\supset \bigcap_{n=1}^{\infty} \{x: f_n(x) \geqslant \tfrac{1}{2}\} = \{x: f(x) \geqslant \tfrac{1}{2}\} \\ &\sim \{x: f(x) = 1\} \sim E; \end{aligned}$$

thus, E is equivalent to $\bigcap_{n=1}^{\infty} \{x: f_n(x) > \frac{1}{4}\}$, a set of type G_δ.

Example 13. Let $(I, \boldsymbol{L}, m)$ be the Lebesgue measure space associated with the unit interval. If $\boldsymbol{T}$ is the ordinary topology on I, then one proof of the theorem of Vitali and Carathéodory shows that $(I, \boldsymbol{T}, \boldsymbol{L}, m)$ is an α-space with α not larger than 2. (On the other hand, since the Lebesgue measure is outer regular and because semicontinuous functions are of Baire type 1 in this topology, one notes that Theorem 12 yields the theorem of Vitali and Carathéodory as a simple corollary.) Since there exists a Lebesgue measurable set E such that both E and $I-E$ meet each non-empty element of $\boldsymbol{T}$ in a set of positive measure, it follows from Theorem 8 that $(I, \boldsymbol{T}, \boldsymbol{L}, m)$ is neither a 1-space nor a 0-space. Thus, the class of all 2-spaces is non-empty.

If a measure satisfies the G_δ condition of Theorem 12, then it is said to be *smooth*. The term was introduced by Schaerf [13] in connection with his study of Lusin's theorem in a general setting. (If the discussion be restricted to a consideration of real-valued functions, then one says that Lusin's theorem holds for a topological measure space $(X, \boldsymbol{T}, \Sigma, \mu)$ in case there exists for each measurable (Σ) function f and for each positive number ε, a corresponding measurable set E such that the restriction of f to E is a continuous function, and $\mu(X-E) < \varepsilon$.) The following proposition is a trivial consequence of Theorem 12 (above) and Theorem 1 and its supplement of [13].

COROLLARY 14. *A topological measure space is an α-space, with $\alpha \leqslant 2$, if and only if Lusin's theorem holds in that space. Thus, the generalized Lusin theorem is equivalent to a generalization of the theorem of Vitali and Carathéodory.*

By virtue of Theorems 6, 8 and 12 and the examples accompanying them, the promise made at the beginning of this section has been redeemed.

4. A general classification theorem. The theorems of the last section show that those α-spaces with $\alpha \leqslant 2$ are characterized by certain regularity or smoothness properties of their measures. The similarity of the methods used to prove Theorems 8 and 12 leads one to suspect that those theorems are special cases of a more general result of the same type. This is indeed the case, as the next theorem shows. After first noting a few elementary properties of the classes $\mathscr{L}_\alpha$ and $\mathscr{U}_\alpha$, one can easily construct a proof of the proposition by modifying slightly the argument used to establish Theorem 12.

In order to simplify the statement of the theorem, one finds it convenient to first extend the scheme of notation customarily employed in the canonical classification of the Borel subsets of a topological space. A set is said to be of type G_{-1} (or F_{-1}) if it is both closed and open. If α is an ordinal, then G_α and F_α have their usual meanings. (If $\alpha = 0$, one defines $\alpha - 1$ to be -1.)

THEOREM 15. *If α is a finite ordinal number, then a topological measure space has property P_α if and only if each measurable set is equivalent to a set of type $G_{\alpha-1}$ ($F_{\alpha-1}$).*

Since no α-spaces with $\alpha > 2$ are known to exist, Theorem 15 is somewhat modestly proclaimed.

BIBLIOGRAPHY

[1] H. Blumberg, *The measurable boundaries of an arbitrary function*, Acta Mathematica 65 (1935), p. 263-282.

[2] A. Denjoy, *Sur les fonctions dérivées sommables*, Bulletin de la Société Mathématigue de France 43 (1916), p. 161-248.

[3] J. Dixmier, *Sur certains espaces considérés par M. H. Stone*, Summa Brasil. Math. 2 (1951), p. 151-182.

[4] C. Goffman, *Real Functions*, New York 1953.

[5] C. Goffman, C. J. Neugebauer, and T. Nishiura, *Density topology and approximate continuity*, Duke Mathematical Journal 28 (1961), p. 497-506.

[6] C. Goffman and D. Waterman, *Approximately continuous transformations*, Proceedings of the American Mathematical Society 12 (1961), p. 116-121.

[7] O. Haupt and C. Pauc, *La topologie approximative de Denjoy envisagée comme vraie topologie*, C. R. Acad. Sci. Paris 234 (1952), p. 390-392.

[8] J. L. Kelley, *General Topology*, New York 1955.

[9] E. Marczewski and R. Sikorski, *Remarks on measure and category*, Colloquium Mathematicum 2 (1949), p. 13-19.

[10] Y. Mibu, *Relations between measure and topology in some Boolean space*, Proc. Imp. Acad. Tokyo 20 (1944), p. 454-458.

[11] K. Nagami, *Baire sets, Borel sets and some typical semicontinuous functions*, Nagoya Mathematical Jounal 7 (1954), p. 85-93.

[12] J. C. Oxtoby, *Spaces that admit a category measure*, Journal für reine und angewandte Mathematik 205 (1961), p. 156-170.

[13] H. M. Schaerf, *On the continuity of measurable functions in neighborhood spaces*, Port. Math. 6 (1947), p. 33-44.

[14] M. H. Stone, *Boundedness properties in function lattices*, Canadian Journal of Mathematics 1 (1949), p. 176-186.

[15] Hing Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Mathematical Journal 19 (1952), p. 289-292.

[16] R. E. Zink, *On semicontinuous functions and Baire functions*, Transactions of the American Mathematical Society 117 (1965), p. 1-9.

Reçu par la Rédaction le 2.8. 1963;
en version modifiée le 5. 5. 1964 et 18. 6. 1965