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ON ALMOST EVERYWHERE AND MEAN CONVERGENCE OF HERMITE AND LAGUERRE EXPANSIONS

BY

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1. Introduction. The purpose of this paper is to prove some results concerning almost everywhere convergence for Hermite and Laguerre expansions. We prove certain maximal inequalities associated with the respective Riesz means, which imply the almost everywhere convergence in a routine way. The maximal inequalities are based on certain L^p estimates for the kernel of the Riesz means obtained by adapting a technique used by J. Peetre [4] to deal with constant coefficient elliptic differential operators on $\mathbf{R}^n$. The same technique was used by G. Mauceri in [3] to study the Riesz means for the sublaplacian on the Heisenberg group.

Let $\Phi_\mu(x)$ stand for the normalized n-dimensional Hermite functions and let P_N be the projection of $L^2(\mathbf{R}^n)$ onto the eigenspace spanned by $\{\Phi_\mu : |\mu| = N\}$. The one-dimensional Hermite functions will be denoted by $\varphi_j(t)$. The Riesz means of order α of a function f is defined by

$$S_R^\alpha f(x) = \sum \left(1 - \frac{2N+n}{R}\right)_+^\alpha P_N f(x). \tag{1.1}$$

In [7] we studied the L^p norm and almost everywhere convergence of $S_R^\alpha f$ to f. There we proved that when $\alpha > (n-1)/2$, $n \geq 2$, $S_R^\alpha f$ converges to f in norm for all f in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$. In this paper we give a different proof of this result. We prove

THEOREM 1. *Let $1 \leq p \leq \infty$, $f \in L^p(\mathbf{R}^n)$ and $\alpha > (n-1)/2$. Then we have the uniform estimates*

$$\|S_R^\alpha f\|_p \leq C\|f\|_p.$$

Moreover, $S_R^\alpha f$ converges to f in norm as $R \to \infty$ for f in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$.

In [7] we also proved the almost everywhere convergence of $S_R^\alpha f$ to f when $\alpha > n/2 - 1/3$. In this paper we will improve this result by proving the following.

THEOREM 2. *Assume that $f \in L^p(\mathbf{R}^n)$, $p \geq 2$ and $n \geq 2$. Then we have*

(i) $\lim_{R\to\infty} S_R^\alpha f(x) = f(x)$ *for a.e. x if $\alpha > (n-1)(1/2 - 1/p)$.*

(ii) $\lim_{R\to\infty} S_R^\alpha f(x) = f(x)$ *at every Lebesgue point x of f if $\alpha > (n-1)/2$.*

We can also prove a.e. convergence results for $1 \leq p \leq 2$ but we do not state them here as they are weaker than the results already proved in [7]. The above theorem will be a consequence of the maximal inequality

$$\sup_{R\geq 0} |S_R^\alpha f(x)| \leq C M_p f(x)\,, \tag{1.2}$$

which holds for $\alpha > (n-1)/2$ and will be proved in Section 4.

Next we consider the Laguerre expansions of the following type. Let L_k^{n-1} be the Laguerre polynomials of type $n-1$. Then the functions $c_n\{k!/(k+n-1)!\}^{1/2} L_k^{n-1}(r^2/2)e^{-r^2/4}$ form an orthonormal family in $L^2(\mathbf{R}_+, r^{2n-1}\,dr)$ for a suitable c_n. Given a function f on $\mathbf{R}_+$ we consider the series

$$f(r) = \sum_{k=0}^{\infty} R_k(f) L_k^{n-1}(r^2/2)e^{-r^2/4} \tag{1.3}$$

where $R_k(f)$ is defined by

$$R_k(f) = c_n \frac{k!}{(k+n-1)!} \int_0^\infty f(r) L_k^{n-1}(r^2/2) e^{-r^2/4} r^{2n-1}\,dr\,. \tag{1.4}$$

We studied this series in [6], where we proved the norm convergence of the Riesz means $\sigma_N^\alpha f(r)$ defined by

$$\sigma_N^\alpha f(r) = \sum_{k=0}^{\infty} \left(1 - \frac{2k+n}{N}\right)_+^\alpha R_k(f) L_k^{n-1}(r^2/2)e^{-r^2/4} \tag{1.5}$$

Here we will prove the following.

THEOREM 3. *Assume that $f \in L^p(\mathbf{R}_+, r^{2n-1}\,dr)$, $2 \leq p \leq \infty$. Then $\sigma_N^\alpha f(r) \to f(r)$ a.e. provided $\alpha > (2n-1)(1/2 - 1/p)$.*

Again it is possible to prove a.e. convergence results for $1 \leq p \leq 2$.

The plan of the paper is as follows. In the next section we prove some basic lemmas for Hermite expansions which will then be used in Section 3 to prove the norm estimates for the Riesz kernel. In Section 4 we prove the maximal inequality. For the proof of the almost everywhere convergence results we refer to [2]. In the last section we take up the case of Laguerre expansions.

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2. Some basic lemmas for kernels defined by Hermite functions. We begin with a very simple estimate for the kernel of the partial sum operator S_N. Let $\Phi_k(x,y) = \sum_{|\mu|=k} \Phi_\mu(x)\Phi_\mu(y)$ be the kernel of the projection operator P_k so that the kernel $S_N(x,y)$ of S_N is given by

$$S_N(x,y) = \sum_{k=0}^{N} \Phi_k(x,y). \tag{2.1}$$

LEMMA 2.1. *There is a constant C independent of N, x and y such that for all $x, y \in \mathbf{R}^n$*

$$|S_N(x,y)| \leq CN^{n/2}. \tag{2.2}$$

Proof. This lemma has been proved in [7] using pointwise estimates for the one-dimensional Hermite functions. Here we propose a very simple proof based on Mehler's formula.

In view of the Cauchy–Schwarz inequality it is enough to prove the lemma when $x = y$. Recall that Mehler's formula states that when $|r| < 1$

$$\sum_{k=0}^{\infty} \Phi_k(x,y)r^k = \pi^{-n/2}(1-r^2)^{-n/2} \exp\left(-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2) + \frac{2rx\cdot y}{1-r^2}\right),$$

so that when $x = y$

$$\sum_{k=0}^{\infty} \Phi_k(x,x)r^k = \pi^{-n/2}(1-r^2)^{-n/2} \exp\left(-\frac{1-r}{1+r}|x|^2\right). \tag{2.3}$$

Therefore, for $0 < r < 1$,

$$r^N \sum_{k=0}^{N} \Phi_k(x,x) \leq \sum_{k=0}^{N} r^k \Phi_k(x,x) \leq C(1-r)^{-n/2}$$

where C is independent of N and x. Taking $r = e^{-1/N}$ we obtain

$$e^{-1} S_N(x,x) \leq C(1-e^{-1/N})^{-n/2} \leq CN^{n/2}.$$

This proves Lemma 2.1.

The next lemma is about the kernel $\Phi_k(x,y)$ itself. In view of Lemma 2.1 it is reasonable to conjecture that $|\Phi_N(x,x)| \leq CN^{n/2-1}$; the next lemma shows that this is indeed true.

LEMMA 2.2. *Assume that $n \geq 2$. Then there is a constant C independent of N and x such that*

$$|\Phi_N(x,x)| \leq CN^{n/2-1}. \tag{2.4}$$

We remark in passing that this lemma is not true when $n = 1$. In fact, in that case $\Phi_N(x,x) = (\varphi_N(x))^2$ and the L^∞ norm of $\varphi_N(x)$ behaves like $N^{-1/12}$.

Proof. We prove the lemma by induction. Assume that it is true when $n = m$, $m \geq 2$, and consider the case $n = m+1$. Writing $x = (y,t)$, $y \in \mathbf{R}^m$, $t \in \mathbf{R}$, we have

$$\begin{aligned}\Phi_N(x,x) &= \sum_{|\mu|+j=N} (\Phi_\mu(y))^2(\varphi_j(t))^2 = \sum_{j=0}^{N}(\varphi_j(t))^2\Phi_{N-j}(y,y)\\ &\leq C\sum_{j=0}^{N}(N-j)^{m/2-1}(\varphi_j(t))^2\end{aligned}$$

by the induction hypothesis. By applying Lemma 2.1 with $n = 1$ the above gives the estimate

$$\Phi_N(x,x) \leq CN^{m/2-1}\sum_{j=0}^{N}(\varphi_j(t))^2 \leq CN^{(m+1)/2-1}.$$

So, it suffices to prove the lemma when $n = 2$. Writing $x = (t,s)$ we have

$$\Phi_N(x,x) = \sum_{j=0}^{N}(\varphi_j(t))^2(\varphi_{N-j}(s))^2.$$

In view of (2.3) it is clear that $\Phi_N(x,x)$ is a radial function. Therefore, if $r = |x|$ then we have

$$\Phi_N(x,x) = \sum_{j=0}^{N}(\varphi_j(r))^2(\varphi_{N-j}(0))^2.$$

Now assume that $N = 2m$. Then $\varphi_j(0) = 0$ if j is odd and $(\varphi_{2j}(0))^2$ is explicitly given by (see Szegö [5])

$$(\varphi_{2j}(0))^2 = \frac{\Gamma(2j+1)}{\pi^{1/2}2^{2j}(\Gamma(j+1))^2}.$$

Therefore,

$$\Phi_N(x,x) = \sum_{j=0}^{m}(\varphi_{2j}(r))^2\frac{\Gamma(2m-2j+1)}{\pi^{1/2}2^{N-2j}(\Gamma(m-j+1))^2}.$$

Using Stirling's formula for the gamma function we have

$$\Phi_N(x,x) \leq C\sum_{j=0}^{m}(\varphi_{2j}(r))^2(N-2j+1)^{-1/2}. \tag{2.5}$$

Next we need the following estimates for the Hermite functions, which follow from the table on p. 700 of Askey–Wainger [1]:

$$(2.6)\qquad \begin{aligned} |\varphi_j(t)| &\le Ce^{-dt^2} && \text{if } t^2 > (2j+1)\,, \\ &\le C(1+2j+1-t^2)^{-1/4} && \text{if } t^2 \le (2j+1)\,. \end{aligned}$$

We split the sum in (2.5) into two parts. Let j_0 be the smallest integer such that $r^2 \le 2j_0 + 1$. The sum

$$\sum_{j=0}^{j_0-1} (\varphi_{2j}(r))^2 (N-2j+1)^{-1/2}$$

is clearly bounded since $(\varphi_{2j}(r))^2 \le Ce^{-2dr^2}$. The other sum

$$\sum_{j=j_0}^{m} (\varphi_{2j}(r))^2 (N-2j+1)^{-1/2} \le C \sum_{j=j_0}^{m} (1+J-r^2)^{-1/2}(N+2-J)^{-1/2}$$

where $J = 2j+1$. Since $r^2 \le 2j_0 + 1 = J_0$, the above sum is bounded by

$$\begin{aligned} \sum_{j=j_0}^{m} &(1+J-J_0)^{-1/2}(N+2-J)^{-1/2} \\ &\le \sum_{j=J_0}^{2m+1} (1+j-J_0)^{-1/2}(N+2-j)^{-1/2} \\ &\le \sum_{j=0}^{2m+1-J_0} (1+j)^{-1/2}(N+2-J_0-j)^{-1/2} \\ &= \sum_{j=0}^{a} (1+j)^{-1/2}(a+1-j)^{-1/2}\,. \end{aligned}$$

Clearly, the last sum is bounded. Similarly for $N = 2m+1$. This proves the lemma.

In the next section, where we are going to estimate the L^2 norm of the kernel $S_R^\alpha(x,y)$, we need to know how the kernel changes when it is multiplied by $(x-y)^\beta$. The following lemma gives a precise expression for $(x-y)^\beta S_R^\alpha(x,y)$. More generally, given a function ψ defined on the nonnegative integers we consider the kernel

$$(2.7)\qquad M_\psi(x,y) = \sum_{N=0}^{\infty} \psi(2N+n)\Phi_N(x,y)$$

and see what happens when $M_\psi(x,y)$ is multiplied by $(x-y)^\beta$.

To state the next lemma we need to introduce some notation. Let Δ be the finite difference operator defined by

$$\Delta\psi(N) = \psi(N+1) - \psi(N) \tag{2.8}$$

and let Δ^k be defined inductively. We set $A_j = -d/dx_j + x_j$, $B_j = -d/dy_j + y_j$ and $(B-A)^\gamma = \prod_{j=1}^n (B_j - A_j)^{\gamma_j}$ for any multiindex γ. By $\Delta^k M_\psi(x,y)$ we denote the kernel $M_{\Delta^k\psi}(x,y)$. With this notation we have the following.

LEMMA 2.3. *For every multiindex β we can write*

$$(x-y)^\beta M_\psi(x,y) = \sum C_{\gamma\delta}(B-A)^\gamma \Delta^{|\delta|} M_\psi(x,y),$$

where the sum is extended over all multiindices γ and δ satisfying $2\delta_j - \gamma_j = \beta_j$, $\delta_j \le \beta_j$.

We proved this lemma in [6]. The proof uses the recursion formula for one-dimensional Hermite functions and the fact that

$$\left(-\frac{d}{dt} + t\right)\varphi_j(t) = (2(j+1))^{1/2}\varphi_{j+1}(t). \tag{2.9}$$

3. An L^2 estimate for the Riesz kernel. To prove the maximal inequality which we need to establish almost everywhere convergence we require the following estimate for the kernel $S_R^\alpha(x,y)$ of the Riesz means S_R^α.

THEOREM 3.1. *Assume that $n \ge 2$. Then there is a constant C such that*

$$\Big(\int_{|x-y|\ge r} |S_R^\alpha(x,y)|^2\,dy\Big)^{1/2} \le CR^{n/4}(1+R^{1/2}r)^{-\alpha-1/2}.$$

Proof. In view of the orthonormality of the Hermite functions $\Phi_\mu(x)$ the square of the L^2 norm of the kernel $S_R^\alpha(x,y)$ equals

$$\sum (1-\nu/R)_+^{2\alpha}\Phi_N(x,x) \tag{3.1}$$

where we have set $\nu = 2N+n$. In view of Lemma 2.1 the above is bounded by $CR^{n/2}$, and so the estimate of the theorem is valid when $R^{1/2}r \le 1$. It is therefore enough to prove the theorem when $R^{1/2}r > 1$.

The proof is based on an old idea of Peetre [4]. We split the kernel into two parts and estimate each part separately. For that purpose we take a C^∞ function W_t such that $W_t(s) = 1$ if $s < 1-t$, $W_t(s) = 0$ if $s > 1-t/2$ and $|W_t^{(k)}(s)| \le C_k t^{-k}$ on $1-t < s < t/2$. Here t is a fixed real number, $0 < t < 1$. With this choice of W_t we define

$$S_{R,2}^\alpha(x,y) = \sum (1-\nu/R)_+^\alpha W_t(\nu/R)\Phi_N(x,y), \tag{3.2}$$

$$S_{R,1}^\alpha(x,y) = S_R^\alpha(x,y) - S_{R,2}^\alpha(x,y). \tag{3.3}$$

LEMMA 3.1. *There is a constant C such that*

$$\int |S^{\alpha}_{R,1}(x,y)|^2\,dy \le Ct^{2\alpha+1}R^{n/2}.$$

Proof. The square of the L^2 norm of $S^{\alpha}_{R,1}(x,\cdot)$ is given by

$$\sum (1-\nu/R)_+^{2\alpha}(1-W_t(\nu/R))^2\Phi_N(x,x). \tag{3.4}$$

Since $1-t<\nu/R$ on the support of $1-W_t(\nu/R)$, we have

$$\int |S^{\alpha}_{R,1}(x,y)|^2\,dy \le Ct^{2\alpha}\sum_{1-t\le\nu/R\le 1}\Phi_N(x,x). \tag{3.5}$$

Using the estimate $\Phi_N(x,y)\le CN^{n/2-1}$ we obtain

$$\int |S^{\alpha}_{R,1}(x,y)|^2\,dy \le Ct^{2\alpha}R^{n/2-1}(Rt). \tag{3.6}$$

This proves the lemma.

LEMMA 3.2. *Assume that m is an integer greater than $2\alpha+1$. Then there is a constant C such that*

$$\int |x-y|^{2m}|S^{\alpha}_{R,2}(x,y)|^2\,dy \le Ct^{2\alpha+1-2m}R^{-m+n/2}.$$

Proof. It is enough to prove the estimate

$$\int |(x-y)^{\beta}S^{\alpha}_{R,2}(x,y)|^2\,dy \le Ct^{2\alpha+1-2m}R^{-m+n/2} \tag{3.7}$$

for every β with $|\beta|=m$. If we set $\psi(N)=(1-N/R)_+^{\alpha}W_t(N/R)$ so that $S^{\alpha}_{R,2}(x,y)=M_{\psi}(x,y)$ then in view of Lemma 2.3 we have

$$(x-y)^{\beta}S^{\alpha}_{R,2}(x,y)=\sum C_{\gamma,\delta}(B-A)^{\gamma}\Delta^{|\delta|}S^{\alpha}_{R,2}(x,y) \tag{3.8}$$

where $2|\delta|-|\gamma|=m$, $|\delta|\le m$. On expanding $(B-A)^{\gamma}$ a typical term of the above sum is

$$\sum \Delta^{|\delta|}\psi(2|\mu|+n)B^{\sigma}\Phi_{\mu}(y)A^{\tau}\Phi_{\nu}(x) \tag{3.9}$$

where $2|\delta|-|\sigma|-|\tau|=m$, $|\delta|\le m$. Using the properties of A and B we see that the square of the L^2 norm of the above sum is bounded by

$$\sum |\Delta^{|\delta|}\psi(2N+n)|^2(2N+n)^{|\sigma|+|\tau|}\Phi_N(x,x). \tag{3.10}$$

Now recall that $\psi(N)=(1-N/R)_+^{\alpha}W_t(N/R)$. The effect of Δ acting on $\psi(N)$ is to bring out the factor $R^{-1}t^{-1}$. This is clear when Δ falls on $W_t(N/R)$ as $|W_t'(s)|\le Ct^{-1}$, and when Δ falls on $(1-N/R)_+^{\alpha}$ it brings out $R^{-1}(1-N/R)^{-1}$, which is bounded by $R^{-1}t^{-1}$ since on the support of W_t, $1-N/R>t/2$.

In estimating the above sum we have to treat two cases separately. When at least one Δ falls on W_t the sum is extended over $1-t<\nu/R<1-t/2$

and it is bounded by

$$(3.11)\qquad R^{-2|\delta|}t^{-2|\delta|+2\alpha}\sum_{1-t\leq\nu/R\leq 1-t/2}(2N+n)^{|\sigma|+|\tau|}\Phi_N(x,x).$$

Using the estimate for $\Phi_N(x,x)$ we immediately see that the above is bounded by

$$R^{-2|\delta|+|\sigma|+|\tau|+n/2-1}t^{-2|\delta|+2\alpha}(Rt).$$

Since $2|\delta|-|\sigma|-|\tau|=m$ and $2\alpha+1-2|\delta|>2\alpha+1-2m$ we get the bound $Ct^{2\alpha+1-2m}R^{-m+n/2}$.

When all the finite differences fall on $(1-\nu/R)_+^\alpha$ the above sum is bounded by

$$(3.12)\qquad R^{-2|\delta|}\sum_{\nu/R\leq 1-t/2}(1-\nu/R)_+^{2\alpha-2|\delta|}(2N+n)^{|\sigma|+|\tau|}\Phi_N(x,x).$$

The sum taken over $1-t\leq\nu/R\leq 1-t/2$ gives the same estimate as before. The remaining part is bounded by the integral

$$R^{-2|\delta|+|\sigma|+|\tau|+n/2}\int_0^{1-t}(1-s)^{2\alpha-2|\delta|}\,ds = R^{-m+n/2}\int_t^1 s^{2\alpha-2|\delta|}\,ds.$$

Since $2|\delta|-|\gamma|=m$, $2|\delta|\geq m$ or $2\alpha-2|\delta|\leq 2\alpha-m$ so that $2\alpha-2|\delta|+1\leq 2\alpha+1-m<0$ by the choice of m. Therefore, the above integral is bounded by $Ct^{2\alpha-2|\delta|+1}\leq Ct^{2\alpha+1-2m}$. This completes the proof of Lemma 3.2.

Now we prove Theorem 3.1. When $|x-y|\geq r$, writing

$$\frac{1}{2}|S_R^\alpha(x,y)|^2\leq|S_{R,1}^\alpha(x,y)|^2+r^{-2m}|x-y|^{2m}|S_{R,2}^\alpha(x,y)|^2$$

we have

$$\int_{|x-y|\geq r}|S_R^\alpha(x,y)|^2\,dy\leq C\{t^{2\alpha+1}R^{n/2}+r^{-2m}R^{-m+n/2}t^{2\alpha+1-2m}\}.$$

The choice $t=(R^{1/2}r)^{-1}$ proves the theorem.

4. The maximal inequality and almost everywhere convergence. As a corollary to Theorem 3.1 we first obtain the following theorem concerning the L^p norm of $S_R^\alpha(x,y)$ when $1\leq p\leq 2$.

THEOREM 4.1. *If* $1\leq p\leq 2$ *and* $\alpha>(n-1)/2$ *then*

$$\left(\int_{|x-y|\geq r}|S_R^\alpha(x,y)|^p\,dy\right)^{1/p}\leq CR^{n/(2q)}(1+R^{1/2}r)^{-\alpha-1/2+n(1/p-1/2)}$$

where $1/p+1/q=1$ *and* C *is independent of* R, r *and* x.

Proof. Consider a partition of $\mathbf{R}^n$ into the dyadic annuli $A_i = \{y : 2^i r \leq |x-y| \leq 2^{i+1} r\}$. Applying Hölder's inequality we obtain

$$(4.1)\quad \Big(\int_{|x-y|\geq r} |S_R^\alpha(x,y)|^p \, dy \Big)^{1/p} \leq \sum_{i=0}^{\infty} |A_i|^{(2-p)/(2p)} \Big(\int_{A_i} |S_R^\alpha(x,y)|^2 \Big)^{1/2}.$$

Since $|A_i| \leq C(2^i r)^n$, using the estimate of Theorem 3.1 we get

$$(4.2)\quad \Big(\int_{|x-y|\geq r} |S_R^\alpha(x,y)|^p \, dy \Big)^{1/p} \leq C \sum_{i=0}^{\infty} \frac{R^{n/4}(2^i r)^{n(1/p-1/2)}}{(1+R^{1/2}2^i r)^{\alpha+1/2}}.$$

The sum on the right-hand side of (4.2) is equal to

$$R^{n/(2q)} \sum_{i=0}^{\infty} (2^i t)^B (1+2^i t)^{-A} = R^{n/(2q)} F(t)$$

where $t = R^{1/2} r$, $A = \alpha + 1/2$ and $B = n(1/p - 1/2)$. Once we know that $F(t) \leq C(1+t)^{B-A}$ we are done. The estimate for $F(t)$ is clearly valid for $t \geq 1$ as $0 \leq B < A$, and for $0 < t < 1$, $F(t)$ is bounded by

$$G(t) = \sum_{i=-\infty}^{\infty} (2^i t)^B (1+2^i t)^{-A},$$

which in turn, being bounded on $[1,2]$ and satisfying $G(2^i t) = G(t)$, is also bounded on $(0,\infty)$.

At this point we are in a position to complete the proof of Theorem 1. Taking $p = 1$ and $r = R^{-1/2}$ in Theorem 4.1 we obtain

$$\int_{|x-y|\geq R^{-1/2}} |S_R^\alpha(x,y)| \, dy \leq C.$$

Combining this with the estimate $|S_R^\alpha(x,y)| \leq C R^{n/2}$ gives

$$(4.3)\quad \int_{\mathbf{R}^n} |S_R^\alpha(x,y)| \, dy \leq C$$

when $\alpha > (n-1)/2$. Similarly we have $\int_{\mathbf{R}^n} |S_R^\alpha(x,y)| \, dx \leq C$. Hence it follows immediately that

$$\|S_R^\alpha f\|_p \leq C \|f\|_p, \quad 1 \leq p \leq \infty.$$

For $f \in C_0^\infty(\mathbf{R}^n)$, $S_R^\alpha f \to f$ in norm, and hence a density argument shows that $S_R^\alpha f \to f$ in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$.

Now we can state and prove the main maximal inequality for the Riesz means. Before that let us recall the definition of $M_pf(x)$:

$$M_pf(x) = \Big(\sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)|^p\, dy\Big)^{1/p}. \tag{4.5}$$

The proof of the next theorem as well as that of the previous one are taken from [3].

THEOREM 4.2. *Assume that $f \in L^p(\mathbf{R}^n)$, $p \geq 2$ and $\alpha > (n-1)/2$. Then*

$$\sup_{R\geq 0} |S_R^\alpha f(x)| \leq CM_pf(x).$$

Proof. The proof is based on the estimate of the previous theorem. Let f be a function vanishing in $|x-y| \leq r$. Then

$$|S_R^\alpha f(x)| \leq \Big(\int_{|x-y|\geq r} |S_R^\alpha(x,y)|^q\, dy\Big)^{1/q} \|f\|_p \tag{4.6}$$

where $1/p + 1/q = 1$. Since $q \leq 2$ we have

$$\Big(\int_{|x-y|\geq r} |S_R^\alpha(x,y)|^q\, dy\Big)^{1/q} \leq CR^{n/(2p)}(1+R^{1/2}r)^{-\alpha-1/2+n(1/q-1/2)}.$$

Given f we set $f_k(y) = f(y)$ if $2^k \leq |x-y| \leq 2^{k+1}$ and $f_k(y) = 0$ otherwise. Then

$$\begin{aligned} |S_R^\alpha f(x)| &\leq \sum_{k=-\infty}^{\infty} |S_R^\alpha f_k(x)| \\ &\leq CR^{n/(2p)} \sum_{k=-\infty}^{\infty} (1+R^{1/2}2^k)^{-\alpha-1/2+n(1/q-1/2)} \|f_k\|_p. \end{aligned}$$

Since $\|f_k\|_p \leq C(2^k)^{n/p} M_pf(x)$ the above sum is dominated by

$$\begin{aligned} CM_pf(x) &\sum_{k=-\infty}^{\infty} R^{n/(2p)} 2^{kn/p} (1+R^{1/2}2^k)^{-\alpha-1/2+n(1/q-1/2)} \\ &\leq CM_pf(x) \sum_{k=-\infty}^{\infty} (R^{1/2}2^k)^{n/p}(1+R^{1/2}2^k)^{-\alpha-1/2+n(1/q-1/2)} \\ &= CM_pf(x)G(R^{1/2}). \end{aligned}$$

The function $G(t) = \sum_{k=-\infty}^{\infty}(2^k t)^{n/p}(1+2^k t)^{-\alpha-1/2+n(1/q-1/2)}$ is clearly locally bounded and since $G(2^i t) = G(t)$ it is bounded on $(0,\infty)$. Hence we have $|S_R^\alpha f(x)| \leq CM_pf(x)$ and this proves the theorem.

The deduction of the almost everywhere convergence and convergence at the Lebesgue points are rather routine; we refer to [3] and [4].

Finally, we remark that it is possible to get an L^∞ estimate for $S_R^\alpha(x,y)$ when $|x-y| \geq r$, and hence an estimate for $\int_{|x-y|\geq r} |S_R^\alpha(x,y)|^p\,dy$ when $p \geq 2$. This leads to a.e. convergence for $1 \leq p \leq 2$ as in [3]; we will not pursue this either.

5. Almost everywhere convergence of Laguerre expansions. As we have already observed in [8], there is a close connection between the Laguerre expansions which we want to study and the Weyl multipliers. So we will first study certain maximal operators defined by Weyl multipliers and then deduce results about Laguerre expansions.

Let $\varphi_N^{n-1}(z) = L_N^{n-1}(|z|^2/2)e^{-|z|^2/4}$ and let Q_N be the projection defined by $Q_N f = \varphi_N^{n-1} \times f$. Here the *twisted convolution* of two functions is defined by

$$f \times g(z) = \int_{\mathbf{C}^n} f(z-w)g(w)e^{i\,\mathrm{Im}\, z\cdot\bar{w}/2}\,dw. \tag{5.1}$$

Consider the following Riesz means $\sigma_N^\alpha f$ for functions defined on $\mathbf{C}^n$:

$$\sigma_N^\alpha f(z) = \sum \left(1 - \frac{2k+n}{N}\right)_+^\alpha Q_k f(z), \tag{5.2}$$

which is given by twisted convolution with the kernel

$$\sigma_N^\alpha(z) = \sum \left(1 - \frac{2k+n}{N}\right)_+^\alpha \varphi_k^{n-1}(z). \tag{5.3}$$

The basic maximal inequality we are going to prove is the following.

THEOREM 5.1. *Assume that $p \geq 2$. Then for $\alpha > n - 1/2$*

$$\sup_{N\geq 0} |\sigma_N^\alpha f(z)| \leq C M_p f(z) \tag{5.4}$$

for all functions f in $L^p(\mathbf{C}^n)$.

As in the case of Hermite expansions this theorem will be proved using the following estimate.

THEOREM 5.2. *Assume $1 \leq p \leq 2$. Then for $\alpha > n - 1/2$*

$$\left(\int_{|z|\geq r} |\sigma_N^\alpha(z)|^p\,dz\right)^{1/p} \leq C N^{n/q}(1 + N^{1/2} r)^{-\alpha - 1/2 + 2n(1/p - 1/2)}$$

where $1/p + 1/q = 1$.

To prove this theorem it is enough to prove the following.

LEMMA 5.1.

$$\Big(\int_{|z|\geq r} |\sigma_N^\alpha(z)|^2\,dz\Big)^{1/2} \leq CN^{n/2}(1+N^{1/2}r)^{-\alpha-1/2}.$$

To prove this lemma we need the following estimates.

LEMMA 5.2.

(i)
$$\Big(\int_{\mathbf{C}^n} |\varphi_N^{n-1}(z)|^2\,dz\Big)^{1/2} \leq CN^{(n-1)/2},$$

(ii)
$$\Big(\int_{\mathbf{C}^n} |\varphi_N^n(z)|^2\,dz\Big)^{1/2} \leq CN^{n/2}.$$

Proof. In view of the formula $\sum_{k=0}^N L_k^\alpha(r) = L_N^{\alpha+1}(r)$ we can write $\varphi_N^n(z) = \sum_{k=0}^N \varphi_k^{n-1}(z)$ and since φ_N^{n-1} are orthogonal functions on $\mathbf{C}^n$ we get

$$\int_{\mathbf{C}^n} |\varphi_N^n(z)|^2\,dz = \sum_{k=0}^N \int |\varphi_k^{n-1}(z)|^2\,dz.$$

Therefore, it is enough to prove (i). But (i) is one of the basic properties of Laguerre polynomials.

As before, choosing W_t we split the kernel $\sigma_N^\alpha(z)$ into two parts. For the first part, using Lemma 5.2 we can prove the estimate

(5.5)
$$\Big(\int_{\mathbf{C}^n} |\sigma_{N,1}^\alpha(z)|^2\,dz\Big)^{1/2} \leq Ct^{\alpha+1/2}N^{n/2}.$$

For the remaining part $\sigma_{N,2}^\alpha(z)$ we need to prove the following estimate.

LEMMA 5.3. *Assume that m is an integer greater than $\alpha+1/2$. Then*

$$\int_{\mathbf{C}^n} |z|^{4m}|\sigma_{N,2}^\alpha(z)|^2 dz \leq CN^{-2m+n}t^{2\alpha+1-4m}.$$

The proof of this lemma requires the following result. Consider a function $M(z)$ of the form

$$M(z) = \sum_{k=0}^\infty \psi(k)\varphi_k^{n-1}(z).$$

Define the operators Δ_+ and Δ_- by

$$\Delta_+(\psi(k)) = \psi(k+1)-\psi(k),$$
$$\Delta_-(\psi(k)) = \psi(k)-\psi(k-1).$$

LEMMA 5.4.

$$\frac{1}{2}|z|^2 M(z) = -\sum_{k=0}^{\infty}(k\Delta_-\Delta_+\psi(k) + n\Delta_-\psi(k))\varphi_k^{n-1}(z).$$

Proof. The proof is elementary. We just use the recursion formula for Laguerre polynomials:

$$(5.6)\quad \begin{aligned} &rL_N^{n-1}(r) \\ &= (2N+n)L_N^{n-1}(r) - (N+1)L_{N+1}^{n-1}(r) - (N+n-1)L_{N-1}^{n-1}(r). \end{aligned}$$

Using this and rearranging we get the lemma.

We make the following observation. The effect of multiplying $\sum_{k=0}^{\infty}\psi(k) \times \varphi_k^{n-1}(z)$ by $|z|^2$ is to change ψ into ψ_1 where $\psi_1(k)$ behaves like $k^{-1}\psi(k)$. Repeated application of this lemma shows that $|z|^{2m}\sum_{k=0}^{\infty}\psi(k)\varphi_k^{n-1}(z)$ is of the form $\sum_{k=0}^{\infty}\psi_m(k)\varphi_k^{n-1}(z)$ where $\psi_m(k)$ behaves like $k^{-m}\psi(k)$. This observation can be used as in the Hermite case to prove Lemma 5.3.

The proof of the maximal inequality (5.4) and the deduction of the following theorem are routine.

THEOREM 5.3. *Suppose that $f \in L^p(\mathbf{C}^n)$, $p \geq 2$. Then*

(i) *$\sigma_N^\alpha f(z) \to f(z)$ a.e. provided $\alpha > (2n-1)(1/2 - 1/p)$.*

(ii) *$\sigma_N^\alpha f(z) \to f(z)$ whenever z is a Lebesgue point provided $\alpha > (2n-1)/2$.*

Now to derive the almost everywhere convergence results for Laguerre expansions we proceed as follows. Let f be a radial function. Then we claim that

$$(5.7)\qquad Q_k f = R_k(f)\varphi_k^{n-1}.$$

To see this we use the fact that the Weyl transform of a radial function f reduces to the Laguerre transform (see [2]):

$$W(f) = \sum_{k=0}^{\infty} R_k(f) P_k$$

where P_k are the projections associated with the Hermite expansions on $\mathbf{R}^n$. We also know that $W(\varphi_k^{n-1}) = P_k$ and $W(f \times g) = W(f)W(g)$. Hence the claim. Thus if f is radial we have

$$(5.8)\qquad \sigma_N^\alpha f(r) = \sum_{k=0}^{\infty}\left(1 - \frac{2k+n}{N}\right)_+^\alpha R_k(f) L_k^{n-1}(r^2/2)e^{-r^2/4},$$

and from Theorem 5.3 we get the almost everywhere convergence result for Laguerre expansions.

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