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FIXED POINTS OF PERIOMORPHISMS

BY

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0. Introduction. The existing fixed point theorems for periomorphisms (periodic homeomorphisms) are concerned mostly with a prime period and with a compact space. It is an unsolved problem whether these results can be extended to arbitrary periods and whether or not compactness is necessary.

In this paper we present a fixed point theorem for periomorphisms of an arbitrary period. We use this theorem to answer a special case of a problem attributed to K. Borsuk: "Does every plane continuum which does not separate the plane have the fixed point property for continuous maps?". We show specifically that Borsuk's problem can be answered affirmatively for the case of periomorphisms of an arbitrary period.

The following have been helpful directly or indirectly: A. Borel, K. Borsuk, H. F. J. Lowig, S. MacLane, P. A. Smith, R. G. Swan and the referee.

1. Preliminaries. By a *periomorphism* we mean a periodic homeomorphism. We use the Čech theories of homology and cohomology with the additive group Z of integers and the modular group Z_p of integers modulo p as the coefficient groups.

LEMMA 1. *If X is a compact set in the Euclidean n-space E^n, then* $H^n(X, Z) = 0 = H_n(X, Z)$.

The following result is known, but its proof seemingly is not available in the literature.

LEMMA 2. *If X is a compact set in the Euclidean 2-space E^2, then $H^1(X, Z)$ is a free abelian group.*

Proof. If we embed X in the 2-sphere S^2, then, by the Alexander duality theorem,

$$H^1(X, Z) = H_0(S^2 - X, Z),$$

where the right-hand side denotes the singular homology group. Now $H_0(S^2 - X, Z)$ is a free abelian group, free on the number of generators which is equal to the number of path components of $S^2 - X$.

PROPOSITION 1. *If X is a compact set in the Euclidean 2-space E^2, then, for any prime p, $H^1(X, Z_p) = 0$ if and only if $H^1(X, Z) = 0$.*

Proof. For a prime number p, let the map $p : Z \to Z$ denote the p-fold multiplication, and let $Z_p = Z/Z_p$.

Then for a compact Hausdorff space the exact sequence

$$0 \to Z \xrightarrow{p} Z \to Z_p \to 0$$

yields the exact sequence

$$\ldots \to H^q(X, Z) \xrightarrow{p} H^q(X, Z) \to H^q(X, Z_p) \to H^{q+1}(X, Z) \xrightarrow{p} \ldots,$$

where q is a non-negative integer. Now since X is compact, then, by Lemma 1, we have the right exact sequence

$$H^1(X, Z) \xrightarrow{p} H^1(X, Z) \to H^1(X, Z_p) \to 0.$$

Therefore,

$$H^1(X, Z_p) = H^1(X, Z)/pH^1(X, Z).$$

But every subgroup of a free group is free, so, from Lemma 2, we have $H^1(X, Z_p) = 0$ if and only if $H^1(X, Z) = 0$.

A compact Hausdorff space X is p-acyclic over the group Z_p, where p is prime, if $H_0(X, Z_p) = Z_p$ and $H_q(X, Z_p) = 0$ for $q > 0$.

PROPOSITION 2. *If X is a compact set in the Euclidean 2-space E^2, then X is p-acyclic over Z_p for a prime p if and only if X is connected and $H^1(X, Z) = 0$.*

Proof. For $X \subseteq E^2$ we have $H^q(X, Z_p) = 0$ for $q > 2$, and $H^2(X, Z_p) = 0$, by Lemma 1. From the duality between cycles and cocycles of a net (cocycles and cycles of a conet), we have $H^q(X, Z_p) = H_q(X, Z_p)$ for $q \geqslant 0$, where Z_p is regarded as a field. Therefore, X is p-acyclic if and only if $H^0(X, Z_p) = Z_p$ and $H^1(X, Z_p) = 0$. The dimension of $H^0(X, Z_p)$ equals the number of components of X, so that $H^0(X, Z_p) = Z_p$ if and only if X is connected. Then, from Lemma 2, we have $H^1(X, Z_p) = 0$ if and only if $H^1(X, Z) = 0$. Thus X is p-acyclic if and only if X is connected and $H^1(X, Z) = 0$.

2. Fixed point theorem. The following result was proved by Smith [2]:

PROPOSITION 3. *If X is a finite-dimensional locally compact Hausdorff space and if X is p-acyclic over Z_p for a prime p, then every periomorphism of X with a period a power of a prime has the fixed point property, and the set of fixed points is p-acyclic.*

We state the fixed point theorem.

THEOREM 1. *If f is a periomorphism with an arbitrary period on a connected, compact set in the Euclidean 2-space E^2, and if $H^1(X, Z) = 0$, then the set of fixed points F of f is a non-empty connected, compact set with $H^1(F, Z) = 0$.*

Proof. We consider first the case where the periomorphism f on the set X has a prime period p. Now, since X is a finite-dimensional connected, compact set, and since by hypothesis $H^1(X, Z) = 0$, it follows from Proposition 2 that X is p-acyclic. Then, from Proposition 3, the fixed point set F is non-empty and is p-acyclic. The set F is obviously compact, and, by Proposition 2, F is connected and $H^1(F, Z) = 0$.

For the case where the period of f is arbitrary, we apply induction on the period. Suppose the assertion is true whenever the period is less than n, where n is not prime. Then a prime number is factorable from n, say $n = pm$, where p is a prime. Now, if f is a periomorphism on X with a period n, then $g = f^m$ is a periomorphism on X with a period p. Therefore, the set G of fixed points of g is a non-empty connected, compact set of E^2 with $H^1(G, Z) = 0$.

We show that $f(G) = G$. Since x belongs to G if and only if $f^m(x) = x$, $f(x)$ belongs to $f(G)$ and is a fixed point of f^m because $f^m(f(x)) = f(f^m(x)) = f(x)$. This shows that $f(G)$ is contained in G. Conversely, we show that if x belongs to G, then x belongs to $f(G)$; that is, $f^{-1}(x)$ belongs to G. Now since $G = \{x \mid f^{-m}(x) = x\}$, it follows from the first part of the proof that $f^{-1}(x)$ belongs to G. Therefore, $f(G) = G$.

Now, if t denotes the restriction of f to G, then t is a periomorphism on G with a period k, where $k \leqslant m < n$. From the induction hypothesis we infer that the set of fixed points T of t is non-empty, connected, compact and $H^1(T, Z) = 0$.

We show that $F = T$. It is clear that F is contained in T. Conversely, if x belongs to T, then $f(x) = x$ which implies that x belongs to F. Therefore, $F = T$.

3. Fixed points of plane continua. For the history of the Borsuk problem the reader may consult the paper by Sieklucki [1] and the book by van der Walt [3].

THEOREM 2. *If X is a compact, connected set of the Euclidean 2-space E^2 which does not separate E^2, then a periomorphism with an arbitrary period on X has the fixed point property.*

Proof. By the Alexander duality theorem, the dimension of $H^1(X, Z)$ plus one is equal to the number of components of $E^2 - X$. Now X does not separate E^2 if and only if $H^1(X, Z) = 0$, so that the assertion follows from Theorem 1.

REFERENCES

[1] K. Sieklucki, *A theorem on fixed points for some plane acyclic continua*, Bulletin de L'Académie Polonaise des Sciences 15 (1967), p. 377-379.

[2] P. A. Smith, *Fixed point theorems for periodic transformations*, American Journal of Mathematics 63 (1941), p. 1-8.

[3] T. van der Walt, *Fixed and almost fixed points*, Mathematisch Centrum, Amsterdam 1963.

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