

**A GENERALIZATION OF RADEMACHER'S THEOREM
ON COMPLETE DIFFERENTIAL**

BY

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1. Introduction.

1.1. Rademacher proved his well-known theorem on total differentiability for the case of a function of two real variables. In his paper he asserts (cf. [3], p. 341) that the theorem carries over at once to functions of several real variables. Inspection of his proof reveals that this is not quite the case. Indeed, for the case of three variables, his proof requires the unlikely condition that if $p = (x_0, y_0, z_0)$ and $q = (x_1, y_1, z_1)$ are in an arbitrary measurable set S , at least one vertex of a box with edges parallel to the coordinate axes and with p and q as vertices of the diagonal (e. g., (x_0, y_1, z_1) or (x_1, y_1, z_0)) must be in Stepanoff's proof [5] of an extension of Rademacher's result also fails to take account of this crucial step. In order to rectify this oversight, we introduce below (see 2.1) the concept of the accessible set and, by Accessibility Theorem (see 2.8), we are able to prove the Rademacher theorem for functions of several variables by modifying Rademacher's proof.

1.2. Notation. Let $p = (p^1, \dots, p^n)$ denote a point of Euclidean n -space R^n . Then, for $n \geq 2$, $\pi_n p$ denotes the projection (p^1, \dots, p^{n-1}) . If S is a subset of R^n , we denote by $S(p^k, \dots, p^n)$, $2 \leq k \leq n$, the $(k-1)$ -dimensional section of S consisting of all points (x^1, \dots, x^{k-1}) in R^{k-1} such that $(x^1, \dots, x^{k-1}, p^k, \dots, p^n)$ is in S . We set

$$|p - q| = \left(\sum_{j=1}^n (p^j - q^j)^2 \right)^{1/2},$$

and let $S \times T$ denote the usual Cartesian product. Let $L_n S$ denote n -dimensional Lebesgue measure of S . The term *measurable* is brief for L_n -measurable. The symbol $A[p, S]$ is defined in 2.1. We use $\nabla f(p)$ for the n -tuple of partial derivatives $(f'_1(p), \dots, f'_n(p))$ so that in the inner product notation

$$\nabla f(p) \cdot (p - q) = \sum_{j=1}^n f'_j(p)(p^j - q^j).$$

2. The accessible set.

2.1. Definition. Let S be any subset of R^n and let $p = (p^1, \dots, p^n)$ be any point of S . The *accessible set* of p , denoted by $A[p, S]$, is S itself if $n = 1$, but for $n \geq 2$ consists of all points $x = (x^1, \dots, x^n)$ of S which satisfy the condition that the points $(x^1, \dots, x^j, p^{j+1}, \dots, p^n)$ be in S , $j = 1, \dots, n-1$.

The following lemma follows easily from the definition by induction:

2.2. LEMMA. *If S is any subset of R^n and p a point in S , the following inductive formula holds (see 1.2 for notation):*

$$A[p, S] = \begin{cases} S & \text{if } n = 1, \\ (A[\pi_n p, S(p^n)] \times R^1) \cap S & \text{if } n \geq 2. \end{cases}$$

2.3. THEOREM. *Let S be an L_n -measurable (Borel) (closed) (open) subset of R^n , and let $p = (p^1, \dots, p^n)$ be a point of S satisfying the condition that, if $n \geq 2$, the sections $S(p^k, \dots, p^n)$ are L_{k-1} -measurable (Borel) (closed) (open) for $k = 2, \dots, n$. Then the accessible set $A[p, S]$ is L_n -measurable (Borel) (closed) (open).*

Proof. We prove the theorem for the L_n -measurable case; the other three cases follow by replacing the epithet " L_n -measurable" by "Borel", "closed", and "open".

The theorem is a consequence of 2.2. It is immediate for $n = 1$. We proceed by induction. Let $n \geq 2$ be any assigned positive integer, and assume the validity of the theorem when $n-1$ replaces n . Let S be any L_n -measurable subset of R^n , and let $p = (p^1, \dots, p^n)$ be a point of S such that $S(p^k, \dots, p^n)$ are L_{k-1} -measurable for $k = 2, \dots, n$. Then $S(p^n)$ is L_{n-1} -measurable and, if $n-1 \geq 2$, its sections $S(p^n)(p^k, \dots, p^{n-1}) = S(p^k, \dots, p^n)$ are L_{k-1} -measurable for $k = 2, \dots, n-1$. Accordingly, by the inductive hypothesis, the accessible set $A[\pi_n p, S(p^n)]$ is L_{n-1} -measurable. Now, in view of closure under finite intersections and Cartesian products, the application of 2.2 completes the proof.

2.4. COROLLARY. *If S is a Borel (closed) (open) subset of R^n , then for every point p in S the accessible set $A[p, S]$ is Borel (closed) (open).*

Proof. This follows from Theorem 2.3, since sections of Borel (closed) (open) sets are always Borel (closed) (open).

2.5. COROLLARY. *If S is an L_n -measurable subset of R^n , then for almost every point p in S the accessible set $A[p, S]$ is L_n -measurable.*

Proof. This follows from Theorem 2.3, since the Fubini Theorem asserts that for almost every point p in S the sections $S(p^k, \dots, p^n)$ are L_{k-1} -measurable, $k = 2, \dots, n$.

The following two lemmas are well known and are easily proved:

2.6. LEMMA. *Let S_1, S_2 be L_n -measurable subsets of R^n . Then a point p of R^n is a point of density of $S_1 \cap S_2$ if and only if p is a point of density of S_1 and S_2 .*

2.7. LEMMA. *Let $p = (p^1, \dots, p^n)$ be a point of a cylindrical subset $S \times R^1$ of R^n , $n \geq 2$, where S is an L_{n-1} -measurable subset of R^{n-1} . Then p is a point of density of $S \times R^1$ if and only if $\pi_n p$ is a point of density of S .*

2.8. ACCESSIBILITY THEOREM. *Let S be an L_n -measurable subset of R^n , and let $p = (p^1, \dots, p^n)$ be a point of S such that, if $n \geq 2$, the sections $S(p^k, \dots, p^n)$, $k = 2, \dots, n$, are L_{k-1} -measurable. Then p is a point of density of the accessible set $A[p, S]$ if and only if (1) p is a point of density of S and (2), if $n \geq 2$, the projections (p^1, \dots, p^{k-1}) are points of density of the corresponding sections $S(p^k, \dots, p^n)$, $k = 2, \dots, n$.*

Proof. The theorem is a consequence of Lemma 2.2 and the well-known Lebesgue Density Theorem. It is obvious for $n = 1$. We proceed by induction.

Let $n \geq 2$ be assigned, and assume the validity of the theorem when $n - 1$ replaces n . Assume that $n \geq 3$, since the case $n = 2$ is a consequence of 2.6 and 2.7, in view of the formula $A[p, S] = (S(p^2) \times R) \cap S$. Let S be an L_n -measurable subset of R^n , and let $p = (p^1, \dots, p^n)$ be a point of S for which each section $S(p^k, \dots, p^n)$ is L_{k-1} -measurable, $k = 2, \dots, n$. Then it follows at once that the section $S(p^n)$ is L_{n-1} -measurable and its sections $S(p^n)(p^k, \dots, p^{n-1})$ are L_{k-1} -measurable, $k = 2, \dots, n - 1$. Thus, by the inductive hypothesis, the projection $\pi_n p$ is a point of density of its accessible set $A[\pi_n p, S(p^n)]$ if and only if $\pi_n p$ is a point of density of $S(p^n)$ and (p^1, \dots, p^{k-1}) is a point of density of $S(p^k, \dots, p^n)$ for $k = 2, \dots, n - 1$. Accordingly, applying Lemma 2.7, it follows that p is a point density of $A[\pi_n p, S(p^n)] \times R^1$ if and only if $\pi_n p$ is a point of density of $S(p^n)$ and (p^1, \dots, p^{k-1}) is a point of density of $S(p^k, \dots, p^n)$, $k = 2, \dots, n - 1$. Therefore, by Lemma 2.6, we infer that p is a point of density of $(A[\pi_n p, S(p^n)] \times R^1) \cap S$ if and only if p is a point of density of S and (p^1, \dots, p^{k-1}) is a point of density of $S(p^k, \dots, p^n)$ for $k = 2, \dots, n$. Thus, in view of Lemma 2.2, the theorem is proved.

2.9. ACCESSIBILITY COROLLARY. *Almost every point of an L_n -measurable set is a point of density of its accessible set.*

Proof. This follows at once from Accessibility Theorem, in view of the well-known sectional density theorem (see, for example, Saks [4], p. 215, for the case $n = 2$), which, in particular, asserts that for almost every point p of an L_n -measurable set S , $n \geq 2$, the projections (p^1, \dots, p^{k-1}) are points of density of the corresponding sections $S(p^k, \dots, p^n)$, $k = 2, \dots, n$, which are L_n -measurable by the Fubini Theorem.

3. The Rademacher Theorem.

3.1. Definition. Let $f: S \rightarrow R$ be a real-valued function on a set S in R^n . Then we say that f is *differentiable* (or has a *total differential*) at $p_0 \in S$ if $\nabla f(p_0)$ exists at p_0 and

$$\lim_{p \rightarrow p_0} \frac{|f(p) - f(p_0) - \nabla f(p_0) \cdot (p - p_0)|}{|p - p_0|} = 0.$$

3.2. LEMMA. Let $f: S \rightarrow R$ be a continuous real-valued function defined on an open bounded set S in R^n . If f has first partial derivatives a. e. on S , then for every $\varepsilon > 0$ and $\gamma > 0$ there exist a closed set $E = E(\varepsilon) \subset S$ and a number $\delta = \delta(\varepsilon, \gamma) > 0$ such that

(i) $L_n(S - E) < \varepsilon$;

(ii) the first partial derivatives of f exist and are continuous at every point p in E , and

$$|f(q) - f(p) - \nabla f(p) \cdot (q - p)| < \gamma |q - p|$$

if $q \in A[p, E]$ and $|q - p| < \delta$.

Proof. By the extended form of Egoroff's Theorem (see Hahn and Rosenthal [1], p. 124), the partial difference quotients converge uniformly to the corresponding partial derivatives on a closed set $E = E(\varepsilon) \subset S$ such that $L_n(S - E) < \varepsilon$. Furthermore, the partial derivatives are continuous on E . Then, for any $\gamma > 0$, there exists $\delta = \delta(\varepsilon, \gamma)$ such that

$$(1) \quad |f(x) - f(p) - f'_j(p)(x^j - p^j)| < \frac{\gamma}{2n} |x^j - p^j|$$

$$\text{if } |x - p| = |x^j - p^j| < \delta \text{ and } x \in E;$$

$$(2) \quad |f'_j(x) - f'_j(p)| < \frac{\gamma}{2n} \quad \text{if } |x - p| < \delta.$$

Let $q \in A[p, E]$. Set $x_0 = p$, $x_n = q$, and $x_j = (q^1, \dots, q^j, p^{j+1}, \dots, p^n)$ for $1 \leq j \leq n-1$. Then $x_j \in E$ for $j = 0, \dots, n$, and if $|q - p| < \delta$,

$$\begin{aligned} (3) \quad & |f(q) - f(p) - \nabla f(p) \cdot (q - p)| \\ & \leq \sum_{j=1}^n |f(x_j) - f(x_{j-1}) - f'_j(x_j)(q^j - p^j)| + \sum_{j=1}^n |f'_j(p) - f'_j(x_j)| \cdot |q^j - p^j| \\ & < \frac{\gamma}{2n} \sum_{j=1}^n (q^j - p^j) + \frac{\gamma}{2n} \sum_{j=1}^n (q^j - p^j) \leq \gamma |q - p|. \end{aligned}$$

With (3) the proof is complete.

3.3. LEMMA. *Let $f: S \rightarrow R$ be a real-valued measurable function defined on a bounded measurable set S in R^n . If f has locally bounded difference quotients a. e. on S , then for every $\varepsilon > 0$ there exist a closed set $F = F(\varepsilon) \subset S$ and numbers $M = M(\varepsilon) > 0$, $\delta(\varepsilon) > 0$ such that*

- (i) $L_n(S - F) < \varepsilon$;
- (ii) f is Lipschitzian on F ;
- (iii) if $p \in F$, $q \in S$, and $|q - p| < \delta$, then $|f(q) - f(p)| \leq M|q - p|$.

Proof. We first select a closed set $E \subset S$ such that f has a locally bounded difference quotient, and hence is continuous at each point of E , and $L_n(S - E) < \varepsilon/2$. Now, for every positive integer k , let E_k denote the set of all points $p \in E$ such that $|f(p)| \leq k$ and $|f(q) - f(p)| \leq k|q - p|$ if $q \in S$ and $|q - p| < 1/k$. Then, clearly, E_k is an ascending sequence of sets, which, by the continuity of f on E , are closed. Since, by hypothesis, f has locally bounded difference quotients at each point of E , we know that $E = \bigcup_{k=1}^{\infty} E_k$, and so $L_n(E - E_K) < \varepsilon/2$ for K large enough. Set $F = E_K$, $M = K$, and $\delta = 1/K$. Clearly, (i) and (iii) of the conclusion hold. To verify (ii) we take any q, p in F and consider two cases. If $|q - p| < 1/K$, then, by (iii),

$$|f(q) - f(p)| \leq K|q - p|.$$

If $|q - p| \geq 1/K$, then

$$|f(q) - f(p)| \leq |f(q)| + |f(p)| \leq 2K = 2K^2(1/K) < 2K^2|q - p|.$$

Thus (ii) holds, where the Lipschitz constant is $2K^2$, and so the proof of the lemma is complete.

3.4. LEMMA. *Let there be given any subset S of R^n and any point p which is a point of (outer) density of S . Then, for each assigned $\eta > 0$, there exists a $\delta = \delta(\eta) > 0$ such that, for any $q \in R^n$ with $|q - p| < \delta$, there corresponds a point q^* in S satisfying the inequality*

$$(4) \quad |q - q^*| < \eta|p - q|.$$

Proof. This is a simple exercise, but may be found implicitly in Rademacher [3].

3.5. RADEMACHER THEOREM. *Let $f: S \rightarrow R$ be a real-valued measurable function defined on a bounded measurable set S in R^n . Then f is differentiable a. e. on S if and only if the difference quotients of f are locally bounded a. e. on S .*

Proof. The necessity is obvious since the difference quotient is locally bounded at every point at which the function is differentiable. To prove the sufficiency suppose that the difference quotients are locally

bounded a. e. on S . Give $\varepsilon > 0$, and let F , M , and $\delta = \delta_1$ be the set and numbers with the properties listed in Lemma 3.3. Since f is Lipschitzian on F , it may be extended to be Lipschitzian on all of E^n and, in particular, on a bounded open set G containing F (see McShane [2]). Then f is of bounded variation with respect to each coordinate variable, and so, by the Lebesgue differentiation theorem, the first partial derivatives of f exist a. e. on G . Give $\gamma > 0$. Then, in view of Lemma 3.2, there exist a set $E \subset G$ and a number $\delta = \delta_2$ such that $L_n(G - E) < \varepsilon$, the first partial derivatives exist and are continuous at every point p in E , and

$$(5) \quad |f(q) - f(p) - \nabla f(p) \cdot (q - p)| \leq \frac{\gamma}{6} |q - p|$$

if $q \in A[p, E]$ and $|q - p| < \delta_2$.

Let $E^* = E \cap F$. Then E^* is a closed subset of S , and $L_n(S - E^*) \leq L_n(S - F) + L_n(G - E) < 2\varepsilon$. Let p be any point of E^* which is a point of density of $A[p, E^*]$. Applying Lemma 3.4, choose $\delta_3 > 0$ so that for every $q \in S$ there exists $q^* \in E^*$ such that

$$(6) \quad |q - q^*| < \frac{\gamma}{3M} |q - p| \quad \text{if } |q - p| < \delta_3.$$

Assume without loss that $\gamma < 3M$. Then $|q - q^*| < |q - p|$ and $|q^* - p| < 2|q - p|$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$, take any $q \in S$ such that $|q - p| < \delta$, and choose $q^* \in A[p, E^*]$ so that (6) holds. Then, in view of (5), (6), and (iii) of Lemma 3.3,

$$\begin{aligned} & |f(q) - f(p) - \nabla f(p) \cdot (q - p)| \\ & \leq |f(q) - f(q^*)| + |f(q^*) - f(p) - \nabla f(p) \cdot (q^* - p)| + |\nabla f(p)| \cdot |q - q^*| \\ & \leq M|q - q^*| + \frac{\gamma}{6}|q^* - p| + M|q - q^*| \\ & < \frac{\gamma}{3}|q - p| + \frac{\gamma}{3}|q - p| + \frac{\gamma}{3}|q - p| = \gamma|q - p|. \end{aligned}$$

Accordingly, in view of Accessibility Corollary (2.9), the proof is complete.

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Reçu par la Rédaction le 24. 7. 1971
