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REARRANGEMENTS OF PERIODIC MULTIPLICATIVE ORTHOGONAL SERIES

BY

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Let $p = (p_1, \ldots, p_k, \ldots)$ be a sequence of integers not less than 2 and put $M_0 = 1$, $M_k = p_1 \ldots p_k$. For every $x \in [0,1)$ we have

$$x = \sum_{k=1}^{\infty} \frac{x_k}{M_k}, \qquad \text{where } x_k \in \{0, 1, \ldots, p_k - 1\}$$

(if this representation is not unique, we take a finite expansion). For $k = 0, 1, \ldots$ let r_k be the function defined by

$$r_k(x) = \exp(2\pi i x_k / p_k). \tag{1}$$

By definition $r_k(x)$ is a step function whose values run through the p_kth roots of unity. On the interval $[0,1)$ there are M_k left-closed intervals of constancy each of length $1/M_k$.

Now let $\{\psi_n\}_{n=0}^{\infty}$ be the set of all finite products of these functions. Each natural number n has a unique representation $n = \sum_{k=0}^{\infty} \alpha_k M_k$, where $\alpha_k \in \{0, 1, \ldots, p_{k+1} - 1\}$. For such $n \geq 1$ we define the function ψ_n ($\psi_0 \equiv 1$) by [4]

$$\psi_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{\alpha_k}. \tag{2}$$

This system is the complete set of characters of the countable direct product of cyclic groups of order p_k, transformed to the unit interval in a measure preserving manner. The system $\{\psi_n\}$ is said to be *bounded* if $\sup_j p_j = K < \infty$.

The purpose of this paper is to obtain a lower estimate of a Weil multiplier for unconditional almost everywhere convergence of series with respect to a bounded $\{\psi_n\}$ system and to establish certain statements concerning upper estimates for a majorant of the partial sums for rearranged $\{\psi_n\}$ systems. For the Walsh system an estimate of a Weil multiplier was established by the author [1]–[3] and independently by Nakata [6].

THEOREM 1. *Let $\{\psi_n\}$ be a bounded multiplicative system and let $W(k)$*

be an increasing sequence of numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{kW(k)} = \infty. \tag{3}$$

Then there exists a sequence $\{a_k\}$ *such that*

$$\sum_{k=1}^{\infty} |a_k|^2 W(k) < \infty \tag{4}$$

and there exists a rearrangement $\sigma(k)$ *of the natural numbers such that the series*

$$\sum_{k=1}^{\infty} a_{\sigma(k)} \psi_{\sigma(k)}(x)$$

diverges almost everywhere.

Proof. We choose a sequence of numbers $\{\mu_n\}$, $\mu_n \downarrow 0$, and an increasing sequence of integers $\{N_\nu\}$ for which (cf. (3))

$$\sum_{n=1}^{\infty} \frac{\mu^2(n)}{W(M_n)} < \infty, \tag{5}$$

$$\sum_{n=N_\nu+1}^{N_{\nu+1}} \frac{\mu(n)}{W(M_n)} \geq \nu. \tag{6}$$

For every $\nu = 1, 2, \ldots$ there exists an increasing sequence of integers $l_n \in [0, N_{\nu+1} - N_\nu)$ $(n = 0, \ldots, 2q(\nu) + 1$ for some $q(\nu))$, and an integer $p(\nu) \in [2, K]$ such that (cf. (6))

$$p_{N_\nu + l_n} = p(\nu), \qquad n = 0, \ldots, 2q(\nu) + 1, \tag{7}$$

$$\sum_{n=0}^{2q(\nu)+1} \frac{\mu(N_\nu + l_n)}{W(M_{N_\nu + l_n})} \geq B(K)\nu, \tag{8}$$

where $B(K)$ is a positive constant.

For every $\nu = 1, 2, \ldots$ we now define subgroups $G_{nm}^{(\nu)}$ of functions $\{\psi_n\}$ and cosets $H_{nm}^{(\nu)}$ of these subgroups. We set (cf. (1))

$$H_{nm}^{(\nu)} = g_{nm}^{(\nu)} G_{nm}^{(\nu)}, \qquad n = 0, \ldots, q(\nu), \; m = 0, \ldots, (p(\nu))^n - 1, \tag{9}$$

where

$$g_{n0}^{(\nu)} = \tau_{N_\nu + l_{2n}} \qquad \text{if } m = 0, \tag{10}$$

$$g_{nm}^{(\nu)} = \tau_{N_\nu + l_{2n}} \tau_{N_1 + l_{2s_1+1}}^{\alpha_1} \cdots \tau_{N_\nu + l_{2s_{h(m)}+1}}^{\alpha_{h(m)}} \tag{11}$$

if $m = \alpha_1(p(\nu))^{s_1} + \alpha_2(p(\nu))^{s_2} + \ldots + \alpha_{h(m)}(p(\nu))^{s_{h(m)}}$, $1 \leq \alpha_j \leq p(\nu) - 1$, $0 \leq s_1 < s_2 < \ldots < s_{h(m)}$. We define the subgroup $G_{nm}^{(\nu)}$ for $n \geq 1$

($G_{00}^{(\nu)} = \{\psi_0\}$) as the direct product of n cyclic groups of order $p(\nu)$ with the generators

$$(12)\qquad g_{nmk}^{(\nu)} = \tau_{N_\nu + l_{2(n-k)}} \tau_{N_\nu + l_{2(s_1-k)+1}}^{\alpha_1} \cdots \tau_{N_\nu + l_{2(s_{h(m)}-k)+1}}^{\alpha_{h(m)}},$$

where $k = 1, \ldots, n$ and we put by convention

$$\tau_{N_\nu + l_{2(s_i-k)+1}} = 1 \quad \text{if } k > s_i.$$

The relations (9)–(12) show that the cosets $H_{nm}^{(\nu)}$ are pairwise disjoint.

For $n = 0, \ldots, q(\nu)$ and $m = 0, \ldots, (p(\nu))^n - 1$ we define the sets $E_m^{(n)}$ as follows. Let $E_0^{(0)} = [0,1]$ and let

$$(13)\qquad E_{p(\nu)m+k}^{(n+1)} = \{\, x \in E_m^{(n)} \,;\, g_{nm}^{(\nu)}(x) = \exp(2\pi i k/p(\nu)) \,\},$$

where $k = 0, \ldots, p(\nu) - 1$. Combining (11) and (13), we obtain

$$(14)\qquad E_m^{(n)} \subset \ldots \subset E_{m_k}^{(n-k)} \subset \ldots \subset E_0^{(0)},$$

where $m_k = [m/(p(\nu))^k]$ and therefore

$$(15)\qquad \operatorname{meas} E_m^{(n)} = (p(\nu))^{-n}.$$

Since every $\psi_k \in H_{nm}^{(\nu)}$ can be represented in the form (cf. (9)–(12))

$$g_{nm}^{(\nu)} (g_{nm1}^{(\nu)})^{\beta_1} (g_{nm2}^{(\nu)})^{\beta_2} \cdots (g_{nmn}^{(\nu)})^{\beta_n},$$

where $0 \le \beta_j \le p(\nu) - 1$, the function ψ_k takes a constant value on the set $E_{p(\nu)m}^{(n+1)}$ (cf. (13), (14)). Hence for some $0 \le \alpha(k) \le p(\nu) - 1$ we have

$$(16)\qquad \psi_k(x) = \exp\left(2\pi i \frac{\alpha(k)}{p(\nu)}\right), \qquad x \in E_{p(\nu)m}^{(n+1)},$$

for $\psi_k \in H_{nm}^{(\nu)}$. Let

$$(17)\qquad A_{nm}^{(\nu)} = \{\, k \,;\, \psi_k \in H_{nm}^{(\nu)} \,\}.$$

We define the coefficients $\{a_k\}$ for $M_{N_\nu} \le k < M_{N_{\nu+1}}$ by

$$(18)\qquad a_k = \begin{cases} \exp\left(-2\pi i \dfrac{\alpha(k)}{p(\nu)}\right) \dfrac{\mu(N_\nu + l_{2n+1})}{(p(\nu))^n W(M_{N_\nu + l_{2n+1}})}, & k \in A_{nm}^{(\nu)}, \\ 0 \quad \text{for all other } k \in [M_{N_\nu}, M_{N_{\nu+1}}). \end{cases}$$

From (5), (9)–(12) and (17), (18) it follows that

$$\sum_{k=0}^{\infty}|a_k|^2W(k)=\sum_{\nu=1}^{\infty}\sum_{n=0}^{q(\nu)}\sum_{m=0}^{(p(\nu))^n-1}\sum_{k\in A_{nm}^{(\nu)}}|a_k|^2W(k)$$
$$\le\sum_{\nu=1}^{\infty}\sum_{n=0}^{q(\nu)}W(M_{N_\nu+l_{2n+1}})\sum_{m=0}^{(p(\nu))^n-1}\sum_{k\in A_{nm}^{(\nu)}}|a_k|^2$$
$$\le\sum_{\nu=1}^{\infty}\sum_{n=0}^{q(\nu)}\frac{\mu^2(N_\nu+l_{2n+1})}{W(M_{N_\nu+l_{2n+1}})}\le\sum_{n=1}^{\infty}\frac{\mu^2(n)}{W(M_n)}<\infty.$$

Hence the condition (4) is fulfilled.

We define (cf. (17))

$$S_{nm}^{(\nu)}(x)=\sum_{k\in A_{nm}^{(\nu)}}a_k\psi_k(x). \tag{19}$$

It follows from (13), (16)–(18) that

$$S_{nm}^{(\nu)}(x)=\frac{\mu(N_\nu+l_{2n+1})}{W(M_{N_\nu+l_{2n+1}})},\quad x\in E_{p(\nu)m}^{(n+1)}; \tag{20}$$

$$|S_{nm}^{(\nu)}(x)|=\frac{\mu(N_\nu+l_{2n+1})}{W(M_{N_\nu+l_{2n+1}})},\quad x\in E_m^{(n)}; \tag{21}$$

and besides, since (cf. (21))

$$\int_{E_m^{(n)}}(S_{nm}^{(\nu)}(x))^2\,dx=\|S_{nm}^{(\nu)}\|_2^2,$$

we have

$$S_{nm}^{(\nu)}(x)=0\quad\text{if } x\notin E_m^{(n)}. \tag{22}$$

We now define the desired rearrangement $\sigma(k)$. Let $\lambda(n,m)$ be a one-to-one mapping of the set of pairs (n,m), $n=0,\dots,q(\nu)$; $m=0,\dots,(p(\nu))^n-1$, onto the natural numbers in the interval

$$\left[1,\frac{(p(\nu))^{q(\nu)+1}-1}{p(\nu)-1}\right].$$

We choose $\lambda(n,m)$ determined by the following conditions:

$$\lambda(n,m)>\lambda(n,m_1)\quad\text{if } m<m_1; \tag{23}$$

if $s<n$ and $E_m^{(n)}\subseteq E_q^{(s+1)}$, where $q=p(\nu)l+k$, then

$$\lambda(s,l-1)>\lambda(n,m)>\lambda(s,l)\quad\text{for } k=0, \tag{24}$$

$$\lambda(s,l)>\lambda(n,m)>\lambda(s,l+1)\quad\text{for } 1\le k<p(\nu). \tag{25}$$

For every fixed $\nu = 1, 2, \ldots$ we arrange the sums $S_{nm}^{(\nu)}(x)$ so that the numbers $\lambda(n,m)$ increase. The sums $\sum_{M_{N_\nu} \le k < M_{N_{\nu+1}}} a_k \psi_k(x)$ as a whole are arranged so that ν increases.

We define

$$L(x) = \max_{x \in E_{p(\nu)m}^{(n+1)}} \lambda(n,m), \tag{26}$$

$$F_m^{(n)} = \bigcup_{\lambda(s,q) \ge \lambda(n,m)} E_{p(\nu)q}^{(s+1)}. \tag{27}$$

By virtue of (15), (20), (22), we have

$$\begin{aligned} \int_0^1 \sum_{\lambda(n,m) \le L(x)} S_{nm}^{(\nu)}(x)\,dx &= \sum_{1 \le \lambda(n,m) \le \frac{(p(\nu))^{q(\nu)+1}-1}{p(\nu)-1}} \int_{F_m^{(n)}} S_{nm}^{(\nu)}(x)\,dx \\ &= \sum_{n=0}^{q(\nu)} \sum_{m=0}^{(p(\nu))^n - 1} \int_{E_{p(\nu)m}^{(n+1)}} S_{nm}^{(\nu)}(x)\,dx \\ &= \frac{1}{p(\nu)} \sum_{n=0}^{q(\nu)} \frac{\mu(N_\nu + l_{2n+1})}{W(M_{N_\nu + l_{2n+1}})}. \end{aligned} \tag{28}$$

Combining (21) with (22), we get the inequalities (cf. (13))

$$\begin{aligned} 0 \le \sum_{\lambda(n,m) \le L(x)} S_{nm}^{(\nu)}(x) &\le \sum_{n=0}^{q(\nu)} \sum_{m=0}^{(p(\nu))^n - 1} |S_{nm}^{(\nu)}(x)| \\ &\le \sum_{n=0}^{q(\nu)} \frac{\mu(N_\nu + l_{2n+1})}{W(M_{N_\nu + l_{2n+1}})}. \end{aligned} \tag{29}$$

Consequently (cf. (28), (29)), the sets

$$H_\nu = \left\{ x\,;\ \sum_{\lambda(n,m) \le L(x)} S_{nm}^{(\nu)}(x) \ge \frac{1}{2p(\nu)} \sum_{n=0}^{q(\nu)} \frac{\mu(N_\nu + l_{2n+1})}{W(M_{N_\nu + l_{2n+1}})} \right\} \tag{30}$$

satisfy

$$\operatorname{meas} H_\nu \ge 1/(2p(\nu)). \tag{31}$$

Since the sets H_ν are independent (cf. (19), (30)), it follows from (31) that

$$\operatorname{meas}\left(\limsup_{\nu \to \infty} H_\nu\right) = 1. \tag{32}$$

Thus the series

$$\sum_{k=0}^{\infty} a_k \psi_k(x) = \sum_{\nu=1}^{\infty} \sum_{n=0}^{q(\nu)} \sum_{m=0}^{(p(\nu))^n - 1} S_{nm}^{(\nu)}(x)$$

satisfies the condition (4), and after a certain rearrangement of its terms, it diverges almost everywhere (cf. (8), (30), (32)). Theorem 1 is proved.

Now we state certain upper estimates of a majorant of the partial sums for rearranged $\{\psi_n\}$ systems.

THEOREM 2. *Let $\{\psi_k\}$ be a bounded multiplicative system and let $n(x)$ be an integer-valued function such that $\|n\|_\infty \le M_q - 1$ for some index q and*

$$n(x) = \sum_{j=1}^{L} m_j(x), \tag{33}$$

where the functions $m_i(x)$ $(i = 1, \dots, L)$ have disjoint supports and are decreasing on their supports. Then

$$\Big\| \sum_{k=0}^{n(x)} a_k \psi_{\sigma(k)}(x) \Big\|_1 \le B\sqrt{L} \Big(\sum_{k=0}^{M_q - 1} |a_k|^2 \Big)^{1/2} \tag{34}$$

for all rearrangements $\sigma(k)$ of the integers $k = 0, \dots, M_q - 1$ and for each sequence of complex numbers $\{a_k\}$.

Proof. We can assume that $m_i(x)$ are piecewise constant functions with intervals of constancy of length $1/M_q$. Hence, we can use a discrete form of integral in (34):

$$\Big\| \sum_{k=0}^{n(x)} a_k \psi_{\sigma(k)}(x) \Big\|_1 = \frac{1}{M_q} \sum_{l=0}^{M_q - 1} \Big| \sum_{k=0}^{n(l)} a_k \psi_{\sigma(k)}(l/M_q) \Big|. \tag{35}$$

Let

$$S_l = \sum_{k=0}^{n(l)} a_k \psi_{\sigma(k)}(l/M_q). \tag{36}$$

We define four sets of integers l as follows (cf. (36)):

$$\begin{aligned} E_1 &= \{ l;\ \operatorname{Re} S_l \ge 0,\ \operatorname{Im} S_l \ge 0 \}, & E_2 &= \{ l;\ \operatorname{Re} S_l \ge 0,\ \operatorname{Im} S_l < 0 \}, \\ E_3 &= \{ l;\ \operatorname{Re} S_l < 0,\ \operatorname{Im} S_l \ge 0 \}, & E_4 &= \{ l;\ \operatorname{Re} S_l < 0,\ \operatorname{Im} S_l < 0 \}. \end{aligned} \tag{37}$$

Let (cf. (33), (35))

$$A_i = \{ l;\ l \in \operatorname{supp} m_i \}. \tag{38}$$

Since for every $j = 1, \dots, 4;\ i = 1, \dots, L$,

$$\sum_{l \in A_i \cap E_j} |S_l| \le \Big| \sum_{l \in A_i \cap E_j} \operatorname{Re} S_l \Big| + \Big| \sum_{l \in A_i \cap E_j} \operatorname{Im} S_l \Big| \le \sqrt{2} \Big| \sum_{l \in A_i \cap E_j} S_l \Big|,$$

it follows from (35)–(38) that

$$\Big\| \sum_{k=0}^{n(x)} a_k \psi_{\sigma(k)}(x) \Big\|_1 \le \frac{\sqrt{2}}{M_q} \sum_{j=1}^{4} \sum_{i=1}^{L} \Big| \sum_{l \in A_i \cap E_j} S_l \Big|. \tag{39}$$

For every i, j there exists a decreasing sequence of sets of integers $F_{ij}^{(k)}$ such that

$$\sum_{l \in A_i \cap E_j} S_l = \sum_{k=0}^{M_q - 1} a_k \sum_{l \in F_{ij}^{(k)}} \psi_{\sigma(k)}(l/M_q). \tag{40}$$

Combining (39) and (40), we obtain

$$\begin{aligned} \Big\| \sum_{k=0}^{n(x)} a_k \psi_{\sigma(k)}(x) \Big\|_1 & \\ \le \frac{\sqrt{2}}{M_q} \Big(\sum_{k=0}^{M_q-1} |a_k|^2 \Big)^{1/2} & \sum_{j=1}^{4} \sum_{i=1}^{L} \Big(\sum_{k=0}^{M_q-1} \Big(\sum_{l \in F_{ij}^{(k)}} \psi_{\sigma(k)}(l/M_q) \Big)^2 \Big)^{1/2} \\ \le \frac{\sqrt{2L}}{M_q} \Big(\sum_{k=0}^{M_q-1} |a_k|^2 \Big)^{1/2} & \sum_{j=1}^{4} \Big(\sum_{i=1}^{L} \sum_{k=0}^{M_q-1} \Big(\sum_{l \in F_{ij}^{(k)}} \psi_{\sigma(k)}(l/M_q) \Big)^2 \Big)^{1/2}. \end{aligned} \tag{41}$$

Let

$$\lambda_{ij}(k) = \max\{ l \in E_j ;\ m_i(l) \ge k \},$$
$$\varepsilon_{ij}^{(l)} = \begin{cases} 1 & \text{if } l \in A_i \cap E_j, \\ 0 & \text{if } l \notin A_i \cap E_j. \end{cases}$$

Applying these definitions and using the fact that the functions $m_i(l)$ are decreasing we have

$$\sum_{l \in F_{ij}^{(k)}} \psi_{\sigma(k)}(l/M_q) = \sum_{l \in F_{ij}^{(k)}} \psi_l^{(1)}(\sigma(k)/M_q) = \sum_{l=0}^{\lambda_{ij}(k)} \varepsilon_{ij}^{(l)} \psi_l^{(1)}(\sigma(k)/M_q), \tag{42}$$

where $\psi_l^{(1)}(x)$ is the multiplicative system formed by the sequence $\{p_{q-i+1}\}$, $i = 1, \dots, q$.

By virtue of the results of Hunt–Taibleson [5] and Gosselin–Young [4], we obtain

$$(43)\quad \sum_{k=0}^{M_q-1}\Big(\sum_{l=0}^{\lambda_{ij}(k)} \varepsilon_{ij}^{(l)}\psi_l^{(1)}(\sigma(k)/M_q)\Big)^2$$
$$= \sum_{k=0}^{M_q-1}\Big(\sum_{l=0}^{\lambda_{ij}(\sigma^{-1}(k))} \varepsilon_{ij}^{(l)}\psi_l^{(1)}(k/M_q)\Big)^2 \le BM_q|A_i\cap E_j|.$$

Thus (cf. (41)–(43))

$$\Big\|\sum_{k=0}^{n(x)} a_k\psi_{\sigma(k)}(x)\Big\|_1 \le B\sqrt{L/M_q}\Big(\sum_{k=0}^{M_q-1}|a_k|^2\Big)^{1/2}\sum_{j=1}^{4}\Big(\sum_{i=1}^{L}|A_i\cap E_j|\Big)^{1/2}$$
$$\le B\sqrt{L}\Big(\sum_{k=0}^{M_q-1}|a_k|^2\Big)^{1/2}.$$

This completes the proof of Theorem 2.

COROLLARY. *If $\{\psi_k\}$ is the Walsh system and $m(x) < 2^N$ is a monotonous function, then*

$$\Big\|\sum_{k=0}^{m(x)} a_k\psi_{\sigma(k)}(x)\Big\|_1 \le B\|(a_k)_k\|_2$$

for all rearrangements $\sigma(k)$ of the integers $0, 1, \ldots, 2^N-1$ and for all sequences of numbers $\{a_k\}$, $k = 0, \ldots, 2^N-1$.

Remark. The monotonicity condition for the functions $m_i(l)$ can be replaced by monotonicity after some rearrangement of the numbers $0 \le l < M_q$ of type described in [4].

In this way the following statement can also be proved.

THEOREM 3. *Let q be an integer and $\sigma(k)$ a rearrangement of the integers $k = 0, \ldots, M_q-1$. Then for each sequence of complex numbers $\{a_k\}$*

$$\max_{\sigma}\Big\|\max_{m<M_q}\Big|\sum_{k=0}^{m} a_k\psi_{\sigma(k)}(x)\Big|\Big\|_1$$
$$\le BM_q^{-1/2}\|(a_k)_k\|_2 \max_{\sigma}\Big\|\max_{m<M_q}\Big|\sum_{k=0}^{m}\psi_{\sigma(k)}^{(1)}(x)\Big|\Big\|_2.$$

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