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SPACES WITH BOOLEAN ASSEMBLIES

BY

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Frames are algebraic analogues of topologies. They are nothing more than complete Heyting algebras and they form a category in a natural way (although not in the obvious way). Each topological space S gives us a frame, namely its topology OS, there are, however, many non-spatial frames.

The initial work on frames was done by Dowker and Papert in [2], and by Isbell in [3]. This was later extended by Macnab in [4] and other places. A survey of frame theory can be found in [5].

Given a frame H there is a canonical way of obtaining a super frame NH together with an embedding $H \to NH$. We call NH the *assembly* of H. The elements of NH are essentially the kernels of frame morphisms with domain H.

Since the assembly NH of a frame H is also a frame, the second assembly N^2H and an embedding $NH \to N^2H$ exist. In fact, for each frame H there is a tower

$$H \to NH \to N^2H \to N^3H \to \ldots$$

of assemblies. It is known that this tower stops if and when a Boolean frame is reached. It is also known that the tower can continue into the transfinite and, in some cases, may never stop.

Beazer and Macnab in [1] have given a characterization of those frames H for which NH is Boolean. In this paper we will look at the topological significance of this result. In particular, we will obtain the following characterization:

THEOREM. *For each T_0-space S, the assembly NOS is Boolean if and only if S is scattered.*

The layout of the paper is as follows.

In Section 1 we give the required topological background. In particular, we decompose "scattered" into its three constituent parts, only two of which are relevant here. In Section 2 we give a very brief survey of frame theory and set down the required preliminary facts. (A more detailed survey of frame theory can be found in [5].)

In Section 3 we look at the assembly NOS of a space S. In particular, we construct a fundamental commuting triangle

$$(\triangle) \qquad \begin{array}{ccc} OS & \to & NOS \\ & \searrow & \downarrow\sigma \\ & & OFS \end{array}$$

where FS is a certain space (the front space) associated with S. The map σ is a surjective frame morphism and is the clue to the analysis of NOS. In Section 4 we use diagram $(\triangle)$ to prove the result stated above (in a slightly more general form). We also characterize those spaces S for which the map σ is an isomorphism. We conclude the paper with examples of spaces S such that the assembly NOS is not Boolean but the second assembly N^2OS is, and examples of spaces S where N^2OS is not Boolean.

1. Topological preliminaries. In this section we gather together all the purely topological material which we use in the paper. In particular, we look at the front topology of a space, and we analyze the notion of a scattered space.

As usual, for each topological space S we let OS be the given topology of S. Occasionally we use CS for the family of closed sets of S. We use $(\cdot)^-$ and $(\cdot)^\circ$ for the closure operation and the interior operation of S, respectively, and $(\cdot)'$ for set-theoretical complementation on S.

The *front topology* of S is the smallest topology on S which includes $OS \cup CS$. We let FS be the set S topologized by the front topology and we call FS the *front space* of S.

For each point p of S the closure p^-, the hull p°, and the monad p^+ of p are defined by

$$p^- = \bigcap\{X \in CS\colon p \in X\}, \qquad p^\circ = \bigcap\{U \in OS\colon p \in U\}, \qquad p^+ = p^\circ \cap p^-.$$

We easily check that

$$\mathscr{B} = \{U \cap p^-\colon\ U \in OS, p \in S\}$$

is a base for FS; moreover,

$$\mathscr{B}_p = \{U \cap p^-\colon\ U \in OS, p \in U\}$$

is an F-open F-neighbourhood system of the point p. Notice also that, for $p \in S$, p^+ is the F-closure, the F-hull, and the F-monad of p.

We say that $p \in S$ is a *T_0-point* if $p^+ = \{p\}$, and we say that p is a *T_B-point* if p^+ is F-open. Thus S is T_0 exactly when each of its points is T_0. Analogously, we say that S is T_B if each of its points is T_B. The significance of this concept is given in the following lemma.

1.1. LEMMA. *A space S is T_B if and only if its front topology OFS is a Boolean algebra.*

Let us now look at scattered spaces and, in particular, at the notion of a relatively isolated point.

It is not hard to see that each scattered space is both T_0 and T_B. The main aim of this section is to produce a property (?) of spaces such that

$$\text{scattered} = T_0 + T_B + (?).$$

In order to do this we must refine the notion of an isolated point.

1.2. Definition. Let S be a space, A a subset of S, and p a point of S. Then p is

(i) an *isolated point*,
(ii) a *detached point*,
(iii) a *loose point*

of A if there is some $U \in OS$ such that

(i) $p \in A \cap U \subseteq \{p\}$,
(ii) $p \in A \cap U \subseteq p^+$,
(iii) $p \in A \cap U \subseteq p^-$,

respectively.

Trivially (since $p \in p^+ \subseteq p^-$), each isolated point of a set is detached, and each detached point is loose.

1.3. LEMMA. *For each point p of a space S the following are equivalent:*

(i) *p is T_B.*
(ii) *p is a detached point of some closed set.*
(iii) *The set p^- has a detached point.*
(iv) *There is some $U \in OS$ such that $p^+ = U \cap p^-$.*

Proof. The three implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iv) $\Rightarrow$ (i) are straightforward, so it is sufficient to show (iii) $\Rightarrow$ (iv).

Suppose that q is a detached point of p^-, so there is some $U \in OS$ with

$$q \in U \cap p^- \subseteq p^+.$$

But then, since $q \in U \subset p^-$, we have $p \in U$ so that $p \in q^+$. This gives $p^+ = q^+$, and hence $U \cap p^- \subseteq p^+$. The converse inclusion is trivial, so we have (iv), as required.

We have introduced the notions of a detached point and a loose point in order to decompose the notion of an isolated point into three constituent parts.

1.4. LEMMA. *Let X be a closed subset of a space S.*

(i) *The isolated points of X are exactly the T_0, detached points of X.*
(ii) *The detached points of X are exactly the T_B, loose points of X.*

Proof. (i) Let p be an isolated point of X so that, for some $U \in OS$, $X \cap U = \{p\}$. Then, since X is closed, $p^+ \subseteq X \cap U$ so that $p^+ = \{p\}$. This shows the p is a T_0-point. The other implications are trivial.

(ii) Trivially, each detached point of X is loose and, by 1.3, is also a T_B-point.

Conversely, suppose that p is a T_B, loose point of X. Then (since p is loose) there is some $U \in OS$ such that $p \in X \cap U \subseteq p^-$, and (since p is T_B) there is some $V \in OS$ with $p^+ = V \cap p^-$. But then

$$p \in X \cap U \cap V \subseteq V \cap p^- = p^+,$$

so the p is a detached point of X, as required.

In Section 4 we will be forced to consider certain subsets of the space S which, as subspaces, are bad, in the sense of the following definition.

1.5. Definition. A closed set of a space is *immoral* if it is the closure of its set of loose points.

Thus a set is *immoral* if it lives off its loose points. In the next two lemmas we show that immorality is concerned with having a smallest F-closed generating set.

1.6. Lemma. *Let S be a space, let $X \in CS$, and let L be the set of loose points of X. If B is an F-closed set such that $B^- = X$, then $L \subseteq B$.*

Proof. Let B be an F-closed set such that $B^- = X$ and consider any $p \in L$. Since p is loose, there is some $U \in OS$ with

$$p \in X \cap U \subseteq p^-.$$

Now, for each $V \in OS$, if $p \in V$, then $p \in U \cap V$ so that (since $p \in L \subseteq X = B^-$) there is some point q with

$$q \in B \cap U \cap V \subseteq X \cap U \subseteq p^-,$$

and hence $V \cap p^-$ meets B. This shows that p is a member of the F-closure of B, and hence, since B is F-closed, $p \in B$, as required.

The next lemma gives us a characterization of immorality alluded to above.

1.7. Lemma. *Let S be a space, $X \in CS$, and let L be the set of loose points of X. The following are equivalent:*

(i) $X = L^-$.

(ii) *There is a smallest F-closed set B such that $B^- = X$.*

(iii) $X = L^-$ *and L is F-closed.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $X = L^-$ and consider the family

$$\mathscr{F} = \{B \in CFS \colon B^- = X\}$$

of F-closed sets. Then $\bigcap\mathscr{F}$ is F-closed and, by 1.6, $L \subseteq \bigcap\mathscr{F}$. But then $(\bigcap\mathscr{F})^- = X$ so that $\bigcap\mathscr{F}$ is the required smallest member of $\mathscr{F}$.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds and let B be the given F-closed set. By 1.6 we have $L \subseteq B$, so it is sufficient to show that $B \subseteq L$.

Let $p \in B$. The set $B \cap p^{-\prime}$ is F-closed and does not contain p, so, by the minimality of B, there is some $U \in OS$ with

$$X \cap U \neq \emptyset, \quad B \cap U \cap p^{-\prime} = \emptyset.$$

We show that $p \in X \cap U \subseteq p^-$, so that $p \in L$, as required.

Consider any point $q \in X \cap U$. Then, for each $V \in OS$ (remembering that $B^- = X$),

$$\begin{aligned} q \in V \Rightarrow q \in U \cap V \Rightarrow U \cap V \text{ meets } B &\Rightarrow (\exists r \in S)[r \in B \cap U \cap V] \\ &\Rightarrow (\exists r \in S)[r \in V \cap p^-] \Rightarrow p \in V, \end{aligned}$$

so that $q \in p^-$. Thus we have $X \cap U \subseteq p^-$.

Now we know there is at least one member q of $X \cap U$, so there is some member q of $U \cap p^-$. Thus $p \in U$ and, consequently (since $p \in B \subseteq X$), $p \in X \cap U$, which completes the proof.

In the following "scattered" is defined in a non-standard way; however (as we see later), this definition is equivalent to the usual one. We choose to introduce scattered spaces in this way for the sake of uniformity.

1.8. Definition. A space S is

(i) *scattered*,
(ii) *dispersed*,
(iii) *corrupt*,

if each closed set of S is the closure of its set of

(i) isolated points,
(ii) detached points,
(iii) loose points,

respectively.

Notice that a space S is corrupt exactly when each of its closed sets is immoral.

1.9. Theorem. *For each space S the following are equivalent:*

(i) *S is dispersed.*
(ii) *Each non-empty closed set of S has a detached point.*
(iii) *Each non-empty set of S has a detached point.*
(iv) *S is both T_B and corrupt.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds and let A be any non-empty set of S. By (ii), A^- has a detached point p, so there is some $U \in OS$ with

$$p \in A^- \cap U \subseteq p^+.$$

Since $p \in A^- \cap U$, we see that U meets A, and so there is some point q with

$$q \in A \cap U \subseteq A^- \cap U \subseteq p^+.$$

But then $q \in p^+$ so that $q^+ = p^+$, and hence q is a detached point of A, as required.

(iii) $\Rightarrow$ (iv). Suppose that (iii) holds so that, by 1.3, S is T_B.

Now consider any $X \in CS$ and let L be the set of loose points of X. If $L^- \neq X$, then $X \cap L^{-\prime} \neq \emptyset$ so that, by (iii), there is some $p \in S$ and $U \in OS$ with

$$p \in X \cap L^{-\prime} \cap U \subseteq p^+ \subseteq p^-.$$

But then, since $L^{-\prime} \cap U$ is open, $p \in L$, which is a contradiction (since $p \in L^{-\prime}$). Thus $L^- = X$, as required.

(iv) $\Rightarrow$ (i). This holds by 1.4 (ii).

The following theorem, whose proof is similar to that of 1.9, shows that corruptness is the appropriate property (?) mentioned above.

1.10. THEOREM. *A space S is scattered if and only if it is both T_0 and dispersed, that is exactly when it is T_0, T_B, and corrupt.*

Finally, in the section we obtain a refinement of 1.7 which will be required later.

1.11. LEMMA. *Let S be a space, $X \in CS$, and let L be the set of loose points of X. The following are equivalent:*

(i) *$X = L^-$ and $X - L$ is closed.*

(ii) *There exists a smallest open set B such that $X \cup B = S$ and $(X \cap B)^- = X$.*

(iii) *$X = L^-$ and L is F-open.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds and let $B = X' \cup L$, so that B is open. We show that B is the required set.

Trivially, $X \cup B = S$. Also $X \cap B = L$ so that $(X \cap B)^- = X$. Consider now any open set C such that $X \cup C = S$ and $(X \cap C)^- = X$. Then $X \cap C$ is F-clopen so, by 1.6, $L \subseteq X \cap C$, and hence $B \subseteq C$, which gives the required result.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds and let B be the given set. Since $X \cap B$ is F-clopen, it is sufficient to show that $L = X \cap B$.

By 1.6 we have $L \subseteq X \cap B$. Conversely, if $p \in X \cap B$, then

$$X \cup (B \cap p^{-\prime}) = (X \cup B) \cap (X \cup p^{-\prime}) = S \cup S = S$$

and $p \notin B \cap p^{-\prime}$ so that, by the minimality of B,

$$(X \cap B \cap p^{-\prime})^- \neq X.$$

This gives us some $q \in S$ and $U \in OS$ with

$$q \in X \cap U, \quad X \cap B \cap p^{-\prime} \cap U = \emptyset.$$

But now (since $q \in X = (X \cap B)^-$ and $q \in U$) there is some point r with

$$r \in X \cap B \cap U \subseteq p^- .$$

Thus (since $r \in p^-$ and $r \in U$) we have $p \in U$ so that

$$p \in X \cap B \cap U \subseteq p^- ,$$

and hence $p \in L$, as required.

(iii) $\Rightarrow$ (i). We show that, assuming (iii), the set $A = X' \cup L$ is open.

Consider any $p \in A$. If $p \in X'$, then $p \in A^\circ$. If $p \in L$, then, for some $U \in OS$,

$$p \in X \cap U \subseteq p^-$$

and (since L is F-open) there is some $V \in OS$ with

$$p \in V \cap p^- \subseteq L .$$

But then

$$p \in X \cap U \cap V \subseteq L$$

so that

$$p \in U \cap V \subseteq X' \cup L = A$$

and again $p \in A^\circ$, which gives the required result.

2. Frame theoretic preliminaries. In order to describe our results we, of course, need some general frame theoretic background material. In this section we give a brief survey of this material. A more detailed survey (without proofs) can be found in [5]. Some proofs and related results can be found in [1] and [4]. The reader is recommended to consult [5] before reading beyond this section.

We begin with a definition.

2.1. Definition. A *frame* H is a complete lattice H such that for each element a of H and subset X of H the frame distributive law

$$a \wedge \bigvee X = \bigvee \{a \wedge x \colon x \in X\}$$

holds.

We regard a frame as a (generalized) universal algebra with distinguished elements 0, 1 (the bottom and top, respectively), a distinguished binary operation $\wedge$, and a distinguished infinitary operation $\bigvee$. The motivating examples of frames are topologies. Thus for each topological space S the topology OS of S is a frame where $0 = \emptyset$, $1 = S$, $\wedge$ is $\cap$ (set-theoretical intersection) and $\bigvee$ is $\bigcup$ (set-theoretical union).

Each frame carries a Heyting operation $\supset$ defined by

$$a \supset b = \bigvee \{x \in H \colon a \wedge x \leqslant b\}$$

and having the characteristic property

$$a \wedge x \leqslant b \Leftrightarrow x \leqslant a \supset b.$$

In particular, each frame carries a pseudocomplementation operation $(\cdot)^*$ given by

$$a^* = a \supset 0.$$

For a topology OS we easily check that, for $U, V \in OS$,

$$U \supset V = (U' \cup V)^\circ, \qquad U^* = U^{-\prime}.$$

As with every pseudocomplemented lattice we say that an element a of a frame H is *Boolean* if $a \vee a^* = 1$ and *regular* if $a = a^{**}$. The Boolean elements of a topology OS are exactly the clopen sets of S, and the regular elements of OS are exactly the regular open sets of S.

A frame morphism

$$H \xrightarrow{f} K$$

from a frame H to a frame K is a map f from H to K such that

$$f(0) = 0, \qquad f(1) = 1$$

and, for each $x, y \in H$ and $X \subseteq H$,

$$f(x \wedge y) = f(x) \wedge f(y), \qquad f(\bigvee X) = \bigvee f(X).$$

In the category of frames a monomorphism is exactly an injective morphism but an epimorphism is not necessarily surjective (although, of course, each surjective morphism is epic).

We are concerned with the algebraic structure of a certain set of functions on a frame H, namely the nuclei on H.

2.2. Definition. A *nucleus* on a frame H is a map $j: H \to H$ such that the conditions

(i) $x \leqslant j(x)$,

(ii) $j^2(x) = j(x)$,

(iii) $j(x \wedge y) = j(x) \wedge j(y)$

hold for all $x, y \in H$.

As explained in [5], the nuclei on a frame H represent the kernels of the morphisms $H \to K$ wi h domain H.

For each element a of the frame H the maps u_a, v_a, w_a on H given by

$$u_a(x) = a \vee x, \qquad v_a(x) = a \supset x, \qquad w_a(x) = (x \supset a) \supset a$$

are nuclei on H. We will particularly be concerned with the w-nuclei on a topology OS (one of which is the double negation or regularization map $U \mapsto U^{-\prime-\prime} = U^{-\circ}$).

For a topology OS there is a natural family of nuclei which include the u-nuclei and the v-nuclei. For each subset A of S the map $[A]$ on OS given by

$$[A](U) = (A \cup U)^{\circ}$$

is a nucleus of S. We call these the *standard nuclei* of S. Notice that if A is open, then $[A]$ is just u_A, and if A is closed, then $[A]$ is just v_B, where $B = A'$.

2.3. Definition. The *assembly* of a frame H is the set NH of all nuclei on H.

The fundamental result of frame theory is that the assembly of a frame is itself a frame. In fact, we have the following

2.4. THEOREM. *For each frame H the assembly NH of H is itself a frame and the map*

$$H \to NH, \quad a \mapsto u_a,$$

is a mono-epimorphism. This embedding is an isomorphism exactly when H is a Boolean frame.

A brief description of the proof of 2.4 is given in [5] and the details of the proof can be found in [1] and [4], here, however, we will not require these details.

Since the assembly NH of a frame H is itself a frame, it also has an assembly N^2H (which we call the *second assembly* of H). In fact, we can iterate the assembly construction to obtain a tower

$$H \to NH \to N^2H \to N^3H \to \dots$$

of assemblies. This tower will stop (or rather become constant) if and when a Boolean frame is obtained.

The following result, which is due to Beazer and Macnab ([1], Theorem 2), characterizes those frames with Boolean assemblies.

2.5. THEOREM. *For each frame H the following are equivalent:*

(i) *The assembly NH is Boolean.*

(ii) *For each element a of H there is a smallest element b of H such that $a \leqslant b$ and $b \supset a = a$.*

It is the purpose of this paper to give a direct proof of the spatial case of this result (that is the case where H is a topology OS) and, consequently, expose some of the reasons why this result holds.

We will use quite a lot the w-nuclei (as defined above). In particular, we will require some simple fact concerning these nuclei.

2.6. LEMMA. *For each two elements a, x of a frame H and each nucleus j on H the following hold:*

(i) $j \leqslant w_a \Leftrightarrow j(a) = a$.

(ii) $w_a(x) = 1 \Leftrightarrow x \supset a = a$.

2.7. Corollary. *For each $j, k \in NH$, if $k \not\leqslant j$, then there is some $a \in H$ with $j \leqslant w_a$, $k \not\leqslant w_a$.*

We will give the details of the proof of the following lemma.

2.8. Lemma. *Let a be an element of the frame H and let j be a nucleus on H such that $j \leqslant w_a$ and*

$$(\forall x \in H)[w_a(x) = 1 \Rightarrow j(x) = 1].$$

Then, for each $x \in H$, $w_a(x) = j(a \vee x)$.

Proof. We easily check that

$$w_a(a \vee x) = w_a(x),$$

so it is sufficient to show that for $y \geqslant a$ we have $w_a(y) \leqslant j(y)$.

Consider any $y \geqslant a$ and let $w = w_a(y)$, so that

$$y \supset a = w \supset a \leqslant w \supset y.$$

Also, $y \leqslant w \supset y$ so that, with $z = (w \supset y) \supset a$,

$$z \leqslant y \supset a \leqslant w \supset y,$$

and hence

$$z = z \wedge (w \supset y) \leqslant a.$$

This gives $z \supset a = 1$, and hence $w_a(w \supset y) = 1$.

Now $j(w \supset y) = 1$ so that, since $(w \supset y) \wedge w \leqslant y$, we have

$$w \leqslant j(w) = j(w \supset y) \wedge j(w) \leqslant j(y),$$

which is the required result.

3. The assembly of a space. To analyze the assembly NOS of a space S we use the standard nuclei of S. Each subset A of S gives us a standard nucleus $[A]$, to obtain, however, all such nuclei it is sufficient to use only the F-open subsets of S. This result, which is proved in 3.1 below, is due (more or less) to Dowker and Papert ([2], Lemma 14) and Macnab ([4], Lemma 6.6).

Here and later we use $(\cdot)^{\square}$ for the F-interior operation of S.

3.1. Lemma. *For each two subsets A, B of the space S the following hold:*

(i) $[A] = [A^{\square}]$.

(ii) $[A] = [B] \Leftrightarrow A^{\square} = B^{\square}$.

Proof. (i) Since $A^{\square} \subseteq A$, it follows trivially that $[A^{\square}] \leqslant [A]$. Conversely, consider any U, V in OS such that $V \subseteq A \cup U$. Then $V \cap U' \subseteq A$ so that (since $V \cap U'$ is F-open) $V \cap U' \subseteq A^{\square}$, which gives $V \subseteq A^{\square} \cup U$. This shows that

$$(A \cup U)^{\circ} \subseteq (A^{\square} \cup U)^{\circ},$$

which completes the proof of (i).

(ii) Suppose first that $[A] = [B]$. Then, for each $p \in S$,

$$\begin{aligned} p \in A^{\square} &\Rightarrow (\exists U \in OS)[p \in U \cap p^{-} \subseteq A] \\ &\Rightarrow (\exists U \in OS)[p \in U \subseteq A \cup p^{-\prime}] \\ &\Rightarrow p \in [A](p^{-\prime}) \Rightarrow p \in [B](p^{-\prime}) \\ &\Rightarrow p \in B \cup p^{-\prime} \Rightarrow p \in B. \end{aligned}$$

Thus $A^{\square} \subseteq B$, and hence $A^{\square} \subseteq B^{\square}$. A similar argument shows that $B^{\square} \subseteq A^{\square}$ so that $A^{\square} = B^{\square}$, as required.

Conversely, if $A^{\square} = B^{\square}$, then (i) gives

$$[A] = [A^{\square}] = [B^{\square}] = [B],$$

which completes the proof of (ii).

Our main task in this section is to set up the basic commuting triangle ($\triangle$) where $OS \to NOS$ is the canonical mono-epimorphism

$$OS \to NOS, \quad A \mapsto [A],$$

and $OS \to OFS$ is the insertion of OS into OFS (and so is also mono-epic). Notice that since $OS \to NOS$ is epic, there can be at most one fill in morphism σ. We will show that this fill in is provided by the following definition.

3.2. Definition. For each nucleus j of the space S let

$$\sigma(j) = \bigcup \{j(U) - U \colon\ U \in OS\}$$

so that $\sigma(j)$ is an F-open set of S.

Sometimes it is more convenient to use the following description of σ.

3.3. Lemma. *For each nucleus j of the space S and each $p \in S$,*

$$p \in \sigma(j) \Leftrightarrow p \in j(p^{-\prime}).$$

Proof. The implication $\Leftarrow$ is trivial since $p \notin p^{-\prime}$.

Conversely, suppose that $p \in \sigma(j)$, so there is some $U \in OS$ with $p \in j(U)$, $p \notin U$. But then $U \subseteq p^{-\prime}$ so that $p \in j(U) \subseteq j(p^{-\prime})$, as required.

3.4. Corollary. *For each $j, k \in NOS$,*

(i) $\sigma(j \wedge k) = \sigma(j) \cap \sigma(k)$,

(ii) $j \leqslant k \Rightarrow \sigma(j) \subseteq \sigma(k)$.

In the following (in 3.6) we show that σ is a frame morphism and so has a nucleus, which lives on NOS. The following lemma shows that this second nucleus attempts to standardize the members of NOS.

3.5. Lemma. *For each F-open set A of the space S and each nucleus j of S,*

(i) $\sigma([A]) = A$,

(ii) $j \leqslant [A] \Leftrightarrow \sigma(j) \subseteq A$,
(iii) *j is standard* $\Leftrightarrow j = [\sigma(j)]$.

Proof. (i) Using 3.3 we have, for each $p \in S$,

$$\begin{aligned} p \in \sigma([A]) &\Leftrightarrow p \in (A \cup p^{-\prime})^{\circ} \\ &\Leftrightarrow (\exists U \in OS)[p \in U \subseteq A \cup p^{-\prime}] \\ &\Leftrightarrow (\exists U \in OS)[p \in U \cap p^{-} \subseteq A] \Leftrightarrow p \in A^{\square}, \end{aligned}$$

and so the result follows since A is F-open.

(ii) The implication $\Rightarrow$ follows from 3.4 (ii) and from (i).

Conversely, suppose $\sigma(j) \subseteq A$ so that, for each $U \in OS$, $j(U) \subseteq A \cup U$. But then $j(U) \subseteq (A \cup U)^{\circ}$, and hence $j \leqslant [A]$, as required.

(iii) The implication $\Rightarrow$ follows from 3.1 (i) and from (i), and the implication $\Leftarrow$ is trivial.

Using these preliminary lemmas we can now obtain the main result of this section.

3.6. Theorem. *For each space S the map σ (of 3.2) is the unique morphism $NOS \to OFS$ such that ($\triangle$) commutes. Moreover, this morphism is surjective.*

Proof. By 3.4 (i) the map σ is a $\wedge$-morphism, and by 3.5 (i) σ preserves bounds (since the bottom and top of NOS are $[\emptyset]$ and $[S]$, respectively).

Next consider any $J \subseteq NOS$ and let $A = \bigcup \sigma[J]$ so that A is F-open and, by 3.4 (ii), $A \subseteq \sigma(\bigvee J)$. Also, for each $j \in J$, $\sigma(j) \subseteq A$ so that 3.5 (ii) gives $j \leqslant [A]$, and hence $\bigvee J \leqslant [A]$. But then another application of 3.5 (ii) gives $\sigma(\bigvee J) \subseteq A$ so that

$$\sigma(\bigvee J) = \bigcup \sigma[J].$$

This shows that σ is a frame morphism, and 3.5 (i) proves that σ is surjective and that ($\triangle$) commutes. Finally, we note that since $OS \to NOS$ is epic, there is at most one fill in morphism σ, and so the proof is completed.

This morphism σ enables us to compute pseudocomplements in NOS and, in particular, to determine the regular nuclei of S. To do this we use the pseudocomplement operation of OFS.

3.7. Definition. For each F-open set A of the space S let A^* be the *pseudocomplement* of A in OFS, that is, the set-theoretical complement of the F-closure of A.

The following lemma, which explains our choice of notation, generalizes a result of Macnab ([4], Lemma 6.2).

3.8. Lemma. *For each nucleus j of the space S, $j^* = [\sigma(j)^*]$.*

Proof. Let $A = \sigma(j)$. By 3.5 (ii) we have $j \leqslant [A]$ so that

$$j \wedge [A^*] \leqslant [A] \wedge [A^*] = [A \cap A^*] = [\emptyset],$$

and hence $[A^*] \leqslant j^*$.

Conversely, if $j \wedge k = 0$ for $k \in NOS$, then

$$\sigma(j) \cap \sigma(k) = \sigma(j \wedge k) = \emptyset$$

so that $\sigma(k) \subseteq \sigma(j)'$, and hence (since $\sigma(k)$ is F-open) $\sigma(k) \subseteq A^*$. But then

$$k \leqslant [\sigma(k)] \leqslant [A^*].$$

This shows that for $k \in NOS$

$$j \wedge k = 0 \Leftrightarrow k \leqslant [A^*],$$

which gives the required result.

3.9. Corollary. *For each F-open set A, $[A]^* = [A^*]$.*

Finally, in this section we obtain a description of the regular nuclei of S and a partial description of the Boolean nuclei. To do this we use the corresponding results for standard nuclei.

3.10. Lemma. *For each F-open set A of the space S,*

(i) *$[A]$ is regular $\Leftrightarrow$ A is F-regular,*

(ii) *$[A]$ is Boolean $\Rightarrow$ A is F-clopen.*

Proof. (i) The nucleus $[A]$ is regular if and only if

$$[A] = [A]^{**} = [A^{**}]$$

which, by 3.1 (ii), occurs exactly when $A = A^{**}$, i.e., when A is F-regular.

(ii) Suppose that $[A]$ is Boolean so that

$$[A] \vee [A^*] = [A] \vee [A]^* = 1,$$

and hence

$$A \cup A^* = \sigma([A]) \cup \sigma([A^*]) = \sigma([A] \vee [A^*]) = S.$$

Thus, since $A \cap A^* = \emptyset$, we have $A' = A^*$, and so A is F-clopen.

3.11. Corollary. *For each nucleus j of the space S,*

(i) *j is regular $\Leftrightarrow$ j is standard and $\sigma(j)$ is F-regular,*

(ii) *j is Boolean $\Rightarrow$ j is standard and $\sigma(j)$ is F-clopen.*

We do have a partial converse to the implication of 3.10 (ii). We easily check that for each two subsets A, B of S

$$[A] \wedge [B] = [A \cap B].$$

Also, when A is open, we see that

$$[A] \vee [B] = [A \cup B]$$

as follows.

Let $j = [A] \vee [B]$. Then, remembering that A is open, for each $U \in OS$ we have

$$[A \cup B](U) = (A \cup B \cup U)^\circ = \big(B \cup (A \cup U)^\circ\big)^\circ$$
$$= [B]\big([A](U)\big) \subseteq [B]\big(j(U)\big) \subseteq j^2(U) = j(U)$$

so that

$$[A \cup B] \leqslant [A] \vee [B].$$

The converse comparison is trivial, so we have the required equality.

This shows that for each open set A and for each closed set Y

$$[A] \wedge [A'] = [\emptyset], \quad [A] \vee [A'] = [S],$$
$$[Y'] \wedge [Y] = [\emptyset], \quad [Y'] \vee [Y] = [S],$$

so that $[A]$ and $[Y]$ are Boolean nuclei. In particular,

$$[A \cup Y] = [A \vee Y]$$

is a Boolean nucleus.

This observation will be used in the proof of 4.3 below.

4. The main results. We can now quite quickly obtain our main results, but first we need just one more preliminary lemma. This is a spatial version of 2.6.

4.1. LEMMA. *Let A be an open set of the space S and let $X = A'$. The following hold:*

(i) *For each F-closed set B,*

$$[B'] \leqslant w_A \Leftrightarrow (B \cap X)^- = X.$$

(ii) *For each open set B,*

$$w_A(B) = S \Leftrightarrow (B \cap X)^- = X.$$

Proof. For each F-closed set B we have

$$(B \cap X)^- = X \Leftrightarrow (B' \cup A)^\circ = A \Leftrightarrow [B'](A) = A$$

so that (i) follows from 2.6 (i). Furthermore, for each open set B we have

$$B \supset A = (B' \cup A)^\circ$$

so that (ii) follows from 2.6 (ii).

The first result of this section is a characterization of the standard w-nuclei of S. We show that these are in one-one correspondence with the immoral closed sets of S.

4.2. THEOREM. *Let S be a space, $A \in OS$, $X = A'$, and let L be the set of loose points of X. The following are equivalent:*

(i) *w_A is standard.*

(ii) *X is immoral.*

(iii) *L is F-closed and $w_A = [L']$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that w_A is standard, so there is some F-closed set B with $w_A = [B']$. By 4.1 (i) this gives $(B \cap X)^- = X$.

Now consider any F-closed set C such that $C^- = X$. Then (since $C \subseteq X$) 4.1 (i) gives

$$[C'] \leqslant w_A = [B']$$

so that (by 3.4 (ii) and 3.5 (i)) $C' \subseteq B'$, and hence $B \subseteq C$. In particular, with $C = X$ we see that $B \subseteq X$.

This shows that B is the smallest F-closed set such that $B^- = X$, and so, by 1.7, X is immoral.

(ii) $\Rightarrow$ (iii). Suppose that X is immoral; so, by 1.7, $X = L^-$ and L is F-closed. By 4.1 (i) we have $[L'] \leqslant w_A$. We will use 2.8 to show that $w_A = [L']$.

Consider any $U \in OS$ such that $w_A(U) = S$. By 4.1 (ii) we have $(U \cap X)^- = X$, so, by 1.6, $L \subseteq U \cap X$, and hence $L' \cup U = S$, which gives $[L'](U) = S$. Thus, using 2.8, for each $U \in OS$ we have

$$w_A(U) = [L'](A \cup U) = (L' \cup X' \cup U)^\circ = (L' \cup U)^\circ = [L'](U)$$

so that $w_A = [L']$, as required.

The implication (iii) $\Rightarrow$ (i) is trivial.

In the next result we refine 4.2 to characterize the Boolean w-nuclei of S, and so (remembering 1.11) we obtain a direct proof of the spatial case of Lemma 10 in [1].

4.3. Theorem. *Let S be a space, $A \in OS$, $X = A'$, and let L be the set of loose points of X. The following are equivalent:*

(i) *w_A is Boolean.*

(ii) *X is immoral and L is F-open.*

Proof. (i) $\Rightarrow$ (ii). Suppose that w_A is Boolean so that, by 3.11 (ii), there is some F-clopen set K with $w_A = [K]$. But then, by 4.2 and 3.1 (ii), X is immoral and $L' = K$, which gives (ii).

(ii) $\Rightarrow$ (i). Suppose that (ii) holds so that, by 4.2, $w_A = [L']$ and L is F-clopen. Also, by 1.11, the set $Y = X \cap L'$ is closed. But now $L' = A \cup Y$, so, by the remarks at the end of Section 3, $w_A = [A \cup Y]$ and is Boolean, which gives (i).

In the next two results we convert the above-given two local characterizations into global characterizations.

4.4. Theorem. *For each space S the following are equivalent:*

(i) *The morphism σ of $(\triangle)$ is monic (and so is an isomorphism).*

(ii) *Each nucleus of S is standard.*

(iii) *S is corrupt.*

Proof. (i) $\Rightarrow$ (ii). Consider any nucleus j of S and let $A = \sigma(j)$. Then, by 3.5 (i), we have $\sigma(j) = A = \sigma([A])$ so that, when σ is monic, $j = [A]$, which gives the required result.

(ii) $\Rightarrow$ (iii). This follows from 4.2.

(iii) $\Rightarrow$ (i). Suppose that S is corrupt and consider two distinct nuclei j, k of S. By 2.7, j and k are separated by a w-nucleus, so (by symmetry) we may suppose that there is some $A \in OS$ with $j \leqslant w_A$ and $k \nleqslant w_A$. But then, since S is corrupt, 4.2 gives us some F-open set B with $j \leqslant [B]$ and $k \nleqslant [B]$. Now 3.5 (ii) gives $\sigma(j) \subseteq B$, $\sigma(k) \nsubseteq B$ so that $\sigma(j) \neq \sigma(k)$, as required.

The following result is the spatial version of the Beazer and Macnab characterization ([1], Theorem 2).

4.5. THEOREM. *A space has a Boolean assembly if and only if it is dispersed.*

Proof. Suppose first that S is a space with a Boolean assembly. Then each nucleus j of S is Boolean so that, by 4.3, S is corrupt.

Now σ is surjective so that OFS is the image of a Boolean frame NOS, and hence OFS is also Boolean. Thus S is T_B so that, by 1.9, S is dispersed.

Conversely, suppose that S is a dispersed space. Then S is both T_B and corrupt so that NOS is isomorphic to the Boolean frame OFS, and hence NOS is also Boolean, as required.

Using 4.5 we can produce many spaces S such that NOS is Boolean but OS is not. (We simply take a non-discrete scattered space.) Also, using 4.4, we can find spaces S such that N^2OS is Boolean but NOS is not, and spaces S such that N^2OS is not Boolean. We conclude this paper with some such examples.

Let $(S, \leqslant)$ be a Dedekind complete linearly ordered set with a first point. For each $a \in S$ let

$$U(a) = \{x \in S: x < a\}, \quad X(a) = \{x \in S: a \leqslant x\}.$$

We easily check that $\{U(a): a \in S\}$ are the proper open sets of a topology on S (which we call the *associated topology* of S). The non-empty closed sets are exactly the sets $X(a)$ for $a \in S$.

For each $a \in S$ we have $a^- = X(a)$ so that

$$a \in X(a) \cap S \subseteq a^-,$$

and hence a is a loose point of X. Thus (since $a^- = X(a)$) $X(a)$ is immoral, and so the space S is corrupt. Hence, by 4.4, NOS is isomorphic to OFS.

The canonical base for the space FS is the set of all $U(b) \cap X(a)$ for $a, b \in S$. Now, if $b \leqslant a$, then $U(b) \cap X(a)$ is empty, otherwise it is the interval $[a, b)$. Thus OFS is generated by the left closed right open intervals of S.

The space FS is T_0 so that OFS is Boolean if and only if FS is discrete. This occurs exactly when for each $a \in S$ there is some $b \in S$ with $[a, b) = \{a\}$, that is, when each element of S has an immediate successor or, equivalently (since S is Dedekind complete), when S is well ordered.

The space FS is also T_B, so, by 1.10 and 4.5, $N^2OS = NOFS$ is Boolean exactly when FS is scattered. This occurs when for each non--empty set A of S there are $a, b \in S$ with $A \cap [a, b) = \{a\}$.

These remarks enable us to give the following examples.

4.6. Examples. (a) For each of the following order types μ the associated space S of μ has a non-Boolean assembly but does have a Boolean second assembly:

(i) $\mu = 1 + a^*$, where a is an infinite ordinal;

(ii) $\mu = 1 + \omega^* + \omega$;

(iii) $\mu = 1 + \omega^* \omega$.

(b) For each of the following order types μ the associated space of μ has a non-Boolean second assembly:

(iv) $\mu = 1 + \lambda$.

(v) $\mu = 1 + (\omega + 1)\lambda$.

REFERENCES

[1] R. Beazer and D. S. Macnab, *Modal extensions of Heyting algebras*, Colloquium Mathematicum 41 (1979), p. 1-12.

[2] C. H. Dowker and D. Papert, *Quotient frames and subspaces*, Proceedings of the London Mathematical Society 16 (1966), p. 275-296.

[3] J. R. Isbell, *Atomless parts of spaces*, Mathematica Scandinavica 31 (1972), p. 5-32.

[4] D. S. Macnab, *Modal operators on Heyting algebras*, submitted for publication.

[5] H. Simmons, *A framework for topology*, Logic Colloquium 77, North Holland.

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