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H^p SOBOLEV SPACES

BY

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1. Introduction. Let J_α and I_α denote the Bessel potential and Riesz potential operators on $\mathbf{R}^n$ for $\alpha \in \mathbf{R}$, $J_\alpha : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ defined by $(J_\alpha f)^\wedge(\xi) = (1+|\xi|^2)^{-\alpha/2}\hat{f}(\xi)$ and $I_\alpha : \mathcal{S}'/\mathcal{P} \to \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ denotes the space of polynomials, defined by $(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$. If B is any space of tempered distributions, we can define Sobolev spaces based on B using these potential operators, $J_\alpha(B)$ and $I_\alpha(B)$, to be the image of B under J_α and I_α respectively. Roughly speaking, for $\alpha > 0$, $I_\alpha(B)$ is the space of tempered distributions having derivatives of order α in B, and $J_\alpha(B)$ is the space of tempered distributions having derivatives of orders $\le \alpha$ in B, and we refer to these as the *homogeneous* and *non-homogeneous* Sobolev spaces of order α based on B. Of course we will need to know that B is preserved by certain basic singular integral operators in order to have a useful Sobolev theory. While the inhomogeneous Sobolev spaces are usually preferred since they can be defined without factoring out by polynomials, one frequently gets sharper results by using the homogeneous Sobolev spaces.

In this paper we investigate the spaces $I_\alpha(H^p)$, where H^p denotes the Hardy spaces $H^p(\mathbf{R}^n)$ of Fefferman–Stein [FeS] and $0 < p \le 1$. We give $I_\alpha(H^p)$ the quasi-norm $\|I_\alpha f\|_{I_\alpha(H^p)} = \|f\|_{H^p}$ for some choice of quasi-norm on H^p. The H^p spaces form a natural continuation of the L^p spaces to the range $0 < p \le 1$, and so the Sobolev spaces $I_\alpha(H^p)$ are a natural generalization of the homogeneous Sobolev spaces $I_\alpha(L^p)$ to the range $0 < p \le 1$. For $\alpha > n(1/p-1)$ the distributions in $I_\alpha(H^p)$ coincide with locally integrable functions, and we will prove results only in this range. If $n(1/p-1) < \alpha \le n/p$ and q is defined by $1/q = 1/p - \alpha/n$ (note $1 < q \le \infty$) then the fractional integration theorem (see [CT, Part II, Theorem 4.1], [J]) implies that each element in $I_\alpha(H^p)$ (initially defined modulo polynomials) can be identified with a unique function in L^q, with $\|f\|_q \le c\|f\|_{I_\alpha(H^p)}$ (see § 4 for the case $q = \infty$).

The main result is a characterization of the quasi-norm of functions in $I_\alpha(H^p)$ in terms of difference quotients. Let k be an integer, $k > \alpha$, and

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define

$$D_{k,\alpha}(f)(x) = \left(\int_0^\infty \left(\int_B |\Delta_{ry}^k f(x)|\, dy \right)^2 \frac{dr}{r^{1+2\alpha}} \right)^{1/2}$$

where Δ_{ry}^k denotes the kth difference operator (inductively $\Delta_{ry}^k f(x) = \Delta_{ry}^{k-1}(f(x+ry) - f(x)))$ and B denotes the unit ball in $\mathbf{R}^n$ (actually all that is needed is that B be a bounded set with non-empty interior). Also let

$$T_{k,\alpha} f(x) = \left(\int_0^\infty \left(\int_B \left| f(x+ry) - \sum_{|\beta|<k} \frac{(ry)^\beta}{\beta!} \left(\frac{\partial}{\partial x}\right)^\beta f(x) \right| dy \right)^2 \frac{dr}{r^{1+2\alpha}} \right)^{1/2}$$

if $k-1 < \alpha < k$. We proved in [Str] that $f \in I_\alpha(L^p)$ if and only if $D_{k,\alpha} f \in L^p$ for $k > \alpha$ and $1 < p < \infty$.

THEOREM 1.1. *Let* $0 < p \le 1$, *and* $\alpha > n(1/p - 1)$.

(a) *Assume* $k > \alpha$. *Then the estimates*

$$c(p,k,\alpha)\|f\|_{I_\alpha(H^p)} \le \|D_{k,\alpha} f\|_p \le c'(p,k,\alpha)\|f\|_{I_\alpha(H^p)}$$

hold for all $f \in I_\alpha(H^p)$.

(b) *Assume* $k-1 < \alpha < k$. *Then the estimates*

$$c(p,k,\alpha)\|f\|_{I_\alpha(H^p)} \le \|T_{k,\alpha} f\|_p \le c'(p,k,\alpha)\|f\|_{I_\alpha(H^p)}$$

hold for all $f \in I_\alpha(H^p)$.

We will prove this theorem in the next two sections, using the atomic decomposition of H^p, and a general g-function characterization of H^p of Uchiyama [U]. A different proof of part (a) of this theorem is given in the monograph of Triebel [Tr] (the proof is given for inhomogeneous Sobolev spaces based on local h^p spaces, but these are relatively trivial differences).

It is also possible to characterize the $I_\alpha(H^p)$ quasi-norm by other variants of the functionals $D_{k,\alpha}$ and $T_{k,\alpha}$. For example, if $\alpha < 2$ we can use a symmetric second difference $f(x+ry) - 2f(x) + f(x-ry)$ in place of $\Delta_{ry}^2 f(x)$. The proof is pretty much the same.

In connection with this result we mention the following relatively trivial facts which follow from the boundedness of singular integrals on H^p:

a) if $\alpha > 0$ then $J_\alpha(H^p) = I_\alpha(H^p) \cap H^p$;

b) if k is a positive integer then $f \in I_k(H^p)$ if and only if $(\partial/\partial x)^\beta f \in H^p$ for all $|\beta| = k$;

c) if $\alpha > k$ then $f \in I_\alpha(H^p)$ if and only if $(\partial/\partial x)^\beta f \in I_{\alpha-k}(H^p)$ for all $|\beta| = k$.

All these results come with corresponding estimates.

As an application of the theorem we prove that $I_{n/p}(H^p)$ forms an algebra (the case $p = 1$ is due to Dahlberg [D]). Finally, we prove an atomic decomposition for $I_\alpha(H^p)$. For related results see [FJ1], [FJ2].

Trace properties of the spaces $I_\alpha(H^p)$ are discussed in Torchinsky [To]. A recent paper of Dappa and Trebels [DT] gives analogous characterizations for anisotropic Sobolev spaces (extending results of Bagby [B]) in the range $1 < p < \infty$. It seems likely that their results can be extended to the anisotropic H^p spaces of Calderón and Torchinsky [CT].

Note that the estimates in Theorem 1.1 are proved under the assumption that $f \in I_\alpha(H^p)$. It would be very desirable to drop this hypothesis, and to show that $D_{k,\alpha}f \in L^p$ or $T_{k,\alpha}f \in L^p$ already implies $f \in I_\alpha(H^p)$ under the conditions on k, α given in the theorem. But we have not been able to verify this conjecture, or even the weaker conjecture that $D_{k,\alpha}f \in L^p$ and $f \in L^q$ (where $1/q = 1/p-\alpha/n$) implies $f \in I_\alpha(H^p)$. It would seem that all that is required here is a variant of the standard cut-off function techniques, but they are poorly adapted to H^p spaces. Perhaps some of the methods of [FoS] or [FJ1], [FJ2] might be useful. The author is grateful to Richard Bagby for pointing out this problem.

2. Estimates from above. In this section we prove the estimates from above for $\|D_{k,\alpha}f\|_p$ in Theorem 1.1, using the atomic decomposition of H^p. We have to show that if $f = I_\alpha g$ for $g \in H^p$ then $D_{k,\alpha}f \in L^p$ provided $n(1/p-1) < \alpha < k$, and $T_{k,\alpha}f \in L^p$ if also $\alpha > k-1$, with $\|D_{k,\alpha}f\|_{L^p} \leq c\|g\|_{H^p}$ and $\|T_{k,\alpha}f\|_{L^p} \leq c\|g\|_{H^p}$. By the atomic decomposition theorem ([C], [CW], [L]) we can write $g = \sum_{i=1}^\infty \lambda_i a_i$ where $\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p} \approx \|g\|_{H^p}$ and the a_i are (p,q)-atoms (for fixed q, $1 < q \leq \infty$). Recall that such an atom is defined to be a function satisfying

(i) $\operatorname{supp} a \subseteq Q$ a cube,
(ii) $\|a\|_q \leq |Q|^{1/q-1/p}$,
(iii) $\int x^\beta a(x)\,dx = 0$ for $|\beta| \leq n(1/p-1)$.

Write $b_i = I_\alpha a_i$. Then $f = \sum_{i=1}^\infty \lambda_i b_i$ and we have $D_{k,\alpha}f(x) \leq \sum_{i=1}^\infty |\lambda_i| \times D_{k,\alpha}b_i(x)$, hence

$$\|D_{k,\alpha}f\|_{L^p}^p \leq \int \left(\sum |\lambda_i| D_{k,\alpha}b_i(x)\right)^p dx \leq \int \sum |\lambda_i|^p D_{k,\alpha}b_i(x)^p\,dx$$

because the l^1-norm dominates the $l^{1/p}$-norm. A similar estimate holds for $T_{k,\alpha}f$. Thus it suffices to establish

LEMMA 2.1. *Suppose $n(1/p-1) < \alpha < k$, and fix q with $2 \leq q \leq \infty$. Then there is a constant M such that $\|D_{k,\alpha}b\|_{L^p} \leq M$ if $b = I_\alpha a$ and a is a (p,q)-atom. If in addition $\alpha < k-1$ then $\|T_{k,\alpha}b\|_{L^p} \leq M$.*

Proof. Because everything is translation invariant we may assume that

Q is centered at the origin, and a simple dilation argument shows we may also assume $|Q| = 1$. We make the choice (modulo polynomials) of

$$b(x) = c_\alpha \int |x-y|^{\alpha-n} a(y)\, dy$$

for α not an integer $\geq n$ (in that case we have a similar expression with logarithmic terms). Then we have the estimate

$$|(\partial/\partial x)^\beta b(x)| \leq c|x|^{\alpha-n-j-1-|\beta|} \tag{2.1}$$

for $|x| \geq 2$ where $j = [n(1/p-1)]$, provided α is not an integer $\geq n$. Indeed, we have

$$(\partial/\partial x)^\beta b(x) = c \int_Q a(y)[(\partial/\partial x)^\beta |x-y|^{\alpha-n} - T_j(y)]\, dy$$

where $T_j(y)$ is the Taylor expansion of $(\partial/\partial x)^\beta |x-y|^{\alpha-n}$ about the point $y = 0$ of order j, because of property (iii) of atoms. But it is easy to see that the quantity in brackets is $O(|x|^{\alpha-n-j-1-|\beta|})$ as $x \to \infty$ for $y \in Q$. Even if α is an integer $\geq n$, we have the same estimate with logarithmic factors. Finally, it is easy to see that b is locally bounded, so for $\beta = 0$ we have the estimate

$$|b(x)| \leq c(1+|x|)^{\alpha-n-j-1} \tag{2.2}$$

with additional logarithmic factors if α is an integer $\geq n$.

Now to estimate $\|D_{k,\alpha} b\|_{L^p}$ we will use a separate argument for the regions $|x| \leq 4$ and $|x| \geq 4$. For the first case we use the estimate $\|D_{k,\alpha} b\|_2 \leq c\|a\|_2$, which is a trivial exercise using the Plancherel formula (see [Str] for the case $k = 1$) and Hölder's inequality with exponent $2/p > 1$ to obtain

$$\Big(\int_{|x|\leq 4} |D_{k,\alpha} b(x)|^p\, dx \Big)^{1/p} \leq c \Big(\int_{|x|\leq 4} |D_{k,\alpha} b(x)|^2\, dx \Big)^{1/2} \leq c\|a\|_2 \leq M\,.$$

For the second region we break up the r-integration in the definition of $D_{k,\alpha} b(x)$ at the point $r = |x|/2k$. For $r \leq |x|/2k$ we use the mean value theorem to write $\Delta^k_{ry} b(x) = \sum_{|\beta|=k} (ry)^\beta (\partial/\partial x)^\beta b(x')$ for some x' with $|x'| \geq |x|/2$. Then the estimate (2.1) yields $|\Delta^k_{ry} b(x)| \leq cr^k |x|^{\alpha-n-j-1-k}$. Thus

$$\begin{aligned}
&\int_{|x|\geq 4} \Big(\int_0^{|x|/2k} \Big(\int_B |\Delta^k_{ry} b(x)|\, dy \Big)^2 \frac{dr}{r^{1+2\alpha}} \Big)^{p/2} dx \\
&\leq c \int_{|x|\geq 4} \Big(\int_0^{|x|/2k} |x|^{2(\alpha-n-j-1-k)} r^{2(k-\alpha)-1}\, dr \Big)^{p/2} dx \\
&\leq c \int_{|x|\geq 4} |x|^{-p(n+j+1)}\, dx \leq M
\end{aligned} \tag{2.3}$$

since $k - \alpha > 0$ and $p(n + j + 1) > n$.

When $r \geq |x|/2k$ we estimate each of the terms $b(x+rmy)$ for $0 \leq m \leq k$ that enters into $\Delta_{ry}^k b(x)$ separately. The $m = 0$ case is easiest, for by (2.2) we obtain

$$\int\limits_{|x|\geq 4} \Bigg(\int\limits_{|x|/2k}^{\infty} \Big(\int\limits_{B} |b(x)|\, dy \Big)^2 \frac{dr}{r^{1+2\alpha}} \Bigg)^{p/2} dx \leq c \int\limits_{|x|\geq 4} |x|^{-p(n+j+1)}\, dx \leq M\,.$$

To estimate the remaining terms we break up the y-integration into the region $B_1 = \{y \in B: \ |x + rmy| \geq |x|/2\}$ and $B_2 = \{y \in B: \ |x + rmy| \leq |x|/2\}$. For the integral over B_1 we can use the same estimate as above. For the integral over B_2 we use the estimate

$$\begin{aligned}\int\limits_{B_2} |b(x + rmy)|\, dy &\leq c \int\limits_{B_2} (1 + |x + rmy|)^{\alpha-n-j-1}\, dy \\ &\leq cr^{-n} \int\limits_{|u|\leq|x|/2} (1 + |u|)^{\alpha-n-j-1}\, du \\ &\leq cr^{-n}(1 + |x|^{\alpha-j-1})\,,\end{aligned}$$

hence

$$\begin{aligned}&\int\limits_{|x|\geq 4} \Bigg(\int\limits_{|x|/2k}^{\infty} \Big(\int\limits_{B_2} |b(x + rmy)|\, dy \Big)^2 \frac{dr}{r^{1+2\alpha}} \Bigg)^{p/2} dx \\ &\qquad \leq c \int\limits_{|x|\geq 4} \Bigg(\int\limits_{|x|/2k}^{\infty} (1 + |x|^{\alpha-j-1})^2 r^{-2n-1-2\alpha}\, dr \Bigg)^{p/2} dx \\ &\qquad \leq c \int\limits_{|x|\geq 4} \Big(1 + |x|^{\alpha-j-1}\Big)^p |x|^{-p(n+\alpha)}\, dx \leq M\end{aligned}$$

provided $\alpha > n(1/p - 1)$. Summing all the estimates obtained yields $\|D_{k,\alpha}b\|_{L^p} \leq M$.

The proof of the estimate for $T_{k,\alpha}$ in place of $D_{k,\alpha}$ is similar. In the region $|x| \leq 4$ we use the Plancherel formula estimate $\|T_{k,\alpha}b\|_2 \leq c\|a\|_2$ (here we use $\alpha > k - 1$ in order to have

$$\int\limits_0^{\infty} \left| e^{iry\cdot\xi} - \sum_{m=0}^{k-1} \frac{(iry\cdot\xi)^m}{m!} \right|^2 \frac{dr}{r^{1+2\alpha}} = c|\xi|^{-2\alpha})\,.$$

For $|x| \geq 4$ we break up the r-integral at $r = |x|$, and for $r \leq |x|$ we obtain the analogue of (2.3) using Taylor's formula with remainder instead of the mean value theorem to estimate

$$\left| b(x + ry) - \sum_{|\beta|<k} \frac{(ry)^\beta}{\beta!} \left(\frac{\partial}{\partial x}\right)^\beta b(x) \right| \leq cr^k |x|^{\alpha-n-j-1-k}\,.$$

When $r \geq |x|$ we have already estimated the $b(x+ry)$ term, so it remains to estimate

$$\int_{|x|\geq 4} \left(\int_{|x|}^{\infty} \left(\int_B \left| (ry)^{\beta} \left(\frac{\partial}{\partial x}\right)^{\beta} b(x) \right| dy \right)^2 \frac{dr}{r^{1+2\alpha}} \right)^{p/2} dx$$

for $|\beta| \leq k-1$. But since $\alpha > k-1$ we have $|\beta| < \alpha$ so the r-integral converges and by (2.1) we can dominate this by $c\int_{|x|\geq 4} |x|^{-p(n+j+1)}\, dx$.

If α is an integer $\geq n$, the logarithmic factors in (2.1) do not materially alter the proof since we still have $\int_{|x|\geq 4} |x|^{-p(n+j+1)}(\log|x|)^N\, dx \leq M$.

3. Estimates from below. In this section we prove the estimates from below in Theorem 1.1 using the generalized g-function characterization of H^p quasi-norms of Uchiyama [U] (Main Theorem and Example 4). The idea of the proof goes back to Stein [St]. For $\varphi \in \mathcal{S}$ with $\int_{\varphi} = 0$ let

$$g_{\varphi}(f)(x) = \left(\int_0^{\infty} |\varphi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} .$$

Then $\|g_{\varphi}(f)\|_p \approx \|f\|_{H^p}$ provided $\hat{\varphi}$ does not vanish identically on any ray through the origin. Thus it suffices to establish

LEMMA 3.1. *There exists $\varphi \in \mathcal{S}$ with $\int_{\varphi} = 0$ and with $\hat{\varphi}$ not vanishing identically on any ray $\{t\xi : t > 0\}$, $\xi \neq 0$, such that*

$$g_{\varphi}(f)(x) \leq D_{k,\alpha}(F)(x) \tag{3.1}$$

if $F = I_{\alpha} f$ provided $\alpha > n(1/p-1)$ and $k > \alpha$. A similar estimate holds for $T_{k,\alpha}$ if also $\alpha > k-1$.

Proof. We begin by choosing a function $h \in \mathcal{S}$ such that $\hat{h}$ vanishes in a neighborhood of the origin. We then set

$$\psi(x) = \sum_{m=1}^{k} \frac{(-1)^{k-m}}{m^n} \binom{k}{m} h_m(-x) \quad \text{and} \quad \varphi = I_{\alpha}\psi .$$

Because $\hat{\psi}$ vanishes in a neighborhood of the origin we have $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, and it is easy to choose h so that $\hat{\varphi}(t\xi)$ does not vanish identically in t for any $\xi \neq 0$.

Now a simple computation (using $\int h = 0$) shows

$$\varphi_t * f(x) = t^{-\alpha}\psi_t * F(x) = t^{-\alpha} \int \Delta_y^k F(x) h_t(y)\, dy. \tag{3.2}$$

Let A denote the annular region $\{y :\ 1/2 \leq |y| \leq 1\}$. Then we can write

$$\int_{\mathbf{R}^n} f(y)\, dy = c \int_0^{\infty} \int_A f(ry)\, dy\, r^{n-1}\, dr$$

and so

$$|\varphi_t * f(x)| \leq ct^{-\alpha} \int_0^\infty \int_A |\Delta^k_{ry}F(x)||h_t(ry)|\,dy\,r^{n-1}\,dr$$

$$\leq ct^{-\alpha-n} \int_0^\infty \int_A |\Delta^k_{ry}F(x)|\,dy\,H(r/t)r^{n-1}\,dr$$

where we have set $H(t) = \sup_{y\in A}|h(ty)|$. We can write this as

$$t^{-1/2}|\varphi_t * f(x)| \leq \int_0^\infty K(t,r)\Big(r^{-1/2-\alpha}\int_A |\Delta^k_{ry}F(x)|\,dy\Big)\,dr \tag{3.3}$$

where $K(t,r) = t^{-\alpha-n-1/2}r^{n-1/2+\alpha}H(r/t)$. Since

$$g_\varphi(f)(x) = \|t^{-1/2}\varphi_t * f(x)\|_{L^2(dt)}$$

and

$$D_{k,\alpha}(f)(x) \geq \Big\|\int_A |\Delta^k_{ry}F(x)|\,dy\,r^{-1/2-\alpha}\Big\|_{L^2(dr)}$$

we see that (3.3) implies (3.1) provided the integral transform with kernel $K(t,r)$ is bounded on L^2 of $(0,\infty)$. But $K(t,r)$ is homogeneous of degree -1, so the famous Hardy–Littlewood–Pólya theorem applies, and the condition we need to verify is

$$\int_0^\infty K(1,r)r^{-1/2}\,dr = \int_0^\infty r^{n-1+\alpha}H(r)\,dr < \infty. \tag{3.4}$$

But this is obvious since $h \in \mathcal{S}$.

For $T_{k,\alpha}$ the argument is similar. We choose h as before but now set $\psi(x) = h(-x)$, and $\varphi = I_\alpha\psi$. Then

$$\varphi_t * f(x) = t^{-\alpha}\psi_t * F(x) \tag{3.5}$$

$$= t^{-\alpha}\int\Big(F(x+y) - \sum_{|\beta|<k}\frac{y^\beta}{\beta!}\Big(\frac{\partial}{\partial x}\Big)^\beta F(x)\Big)h_t(y)\,dy$$

because the moments of h_t vanish, and this is the analogue of (3.2). The rest of the proof is then the same. ∎

R e m a r k s. The conditions on α are not really required (only $\alpha > -n$ is required in (3.4)). With a little more work we can prove the estimate $g(f)(x) \leq cT_{k,\alpha}(F)(x)$ where $g(f)$ is the usual Littlewood–Paley function. For this we need to take $\hat{h}(\xi) = |\xi|^{1+\alpha}e^{-|\xi|}$ and $\hat{h}(\xi) = \xi_j|\xi|^\alpha e^{-|\xi|}$. The condition $\alpha > k-1$ will imply the vanishing of enough moments to obtain (3.5), and the estimate (3.4) will hold because $h(x) = O(|x|^{-n-\alpha-2})$ as $x \to \infty$. The fact that $g(f) \in L^p$ implies $f \in H^p$ was already established in

Fefferman–Stein [FeS]. However, there is no obvious way to obtain $g(f)(x) \leq cD_{k,\alpha}(F)(x)$.

4. Pointwise multipliers. In the range $n/p \geq \alpha > n(1/p-1)$ we may regard $I_\alpha(H^p)$ as a space of functions in L^q, where $1/q = 1/p - \alpha/n$, and so the operation of multiplication by a function g is well-defined for elements of $I_\alpha(H^p)$. We call g a *multiplier* if this operator is bounded on $I_\alpha(H^p)$. If every element of $I_\alpha(H^p)$ is a multiplier then $I_\alpha(H^p)$ forms an algebra under pointwise multiplication. This happens exactly when $\alpha = n/p$, and it is interesting because the analogous statement is false for $1 < p < \infty$. The case $p = 1$ is proved by Dahlberg [D].

THEOREM 4.1. *$I_{n/p}(H^p)$ is an algebra under pointwise multiplication for $0 < p \leq 1$, and $\|fg\|_{I_{n/p}(H^p)} \leq c\|f\|_{I_{n/p}(H^p)}\|g\|_{I_{n/p}(H^p)}$.*

Proof. Choose $k > n/p$. We will show $D_{2k-1,n/p}(fg) \in L^p$. Now we claim that $I_{n/p}(H^p) \subseteq C_0$ and $\|f\|_\infty \leq c\|f\|_{I_{n/p}(H^p)}$. By the fractional integration theorem it suffices to establish this for $p = 1$, and in this case by the atomic decomposition theorem we may assume f is an atom. Then $I_n f(x) = c_n \int_Q f(y) \log|x-y|\, dy = c_n \int_Q f(y) \log|x-y|/|x|\, dy$. The first expression and the H^1-BMO duality shows that $I_n f$ is continuous and $\|I_n f\|_\infty \leq c$, and the second expression shows $I_n f$ vanishes at infinity.

Now a simple combinatorial identity yields

$$\Delta_{ry}^{2k-1} fg(x) = \sum_{m=0}^{k} \sum_{\ell=0}^{k} (c_{m,\ell}(\Delta_{ry}^k f(x+mry))g(x+\ell ry) + c'_{m,\ell} f(x+\ell ry)\Delta_{ry}^k g(x+mry))$$

for certain coefficients $c_{m,\ell}$ and $c'_{m,\ell}$ whose exact value does not matter for our purposes. It follows that

$$|\Delta_{ry}^{2k-1} fg(x)| \leq c \sum_{m=0}^{k} (\|g\|_\infty |\Delta_{ry}^k f(x+mry)| + \|f\|_\infty |\Delta_{ry}^k g(x+mry)|)$$

and this would yield

$$D_{2k-1,n/p}(fg) \leq \|g\|_\infty D_{k,n/p}(f) + \|f\|_\infty D_{k,n/p}(g)$$

except for the appearance of $x + mry$ in place of x. However, this leads to a harmless variant of $D_{k,n/p}$ and so we conclude

$$\|D_{2k-1,n/p}(fg)\|_{L^p} \leq c\|f\|_{I_{n/p}(H^p)}\|g\|_{I_{n/p}(H^p)}.$$

To complete the proof we need to find a dense subspace H_0^p of H^p such that $I_\alpha f \cdot I_\alpha g$ is a priori in $I_\alpha(H^p)$. For H_0^p we may take the functions f such that $\hat{f} \in \mathcal{D}$ and $\hat{f}$ vanishes in a neighborhood of the origin. The density is proved in [CT, Part II, Theorem 1.8]. Then $I_\alpha f \cdot I_\alpha g$ clearly belongs to $\mathcal{S}$, and $\mathcal{S} \subseteq I_\alpha(H^p)$ for $\alpha > n(1/p-1)$ follows by the same theorem of Calderón and Torchinsky. ■

Remark. If we could prove the conjecture that $D_{k,\alpha}f \in L^p$ and $f \in L^q$ implies $f \in I_\alpha(H^p)$ under the conditions of Theorem 1.1 (a), then we could show that the characteristic function χ_Ω of a Lipschitz domain Ω is a multiplier on $I_\alpha(H^p)$ provided $n(1/p-1) < \alpha < 1/p$. The proof (for $\alpha < 1$) entails showing

$$D_{1,\alpha}(\chi_\Omega)(x) \le c\,\mathrm{dist}(x, \partial\Omega)^{-\alpha}$$

using ideas from [Str]. For related results see [Tr] and [FJ2].

5. Atomic decomposition. Given the atomic decomposition characterization of H^p, it seems natural that there should be an atomic decomposition characterization of $I_\alpha(H^p)$. On a trivial level one only has to apply I_α to H^p atoms, but we can do a little better than that. We again assume $\alpha > n(1/p-1)$. Now this condition has the effect of eliminating the need for moment conditions on the atoms.

DEFINITION 5.1. An *$I_\alpha(p,q)$-atom* (for $2 \le q < \infty$) is a function $b(x)$ supported in a cube Q such that $b \in I_\alpha(L^q)$ and $\|b\|_{I_\alpha(L^q)} \le |Q|^{-1/p}$.

THEOREM 5.2. *Let $\alpha > n(1/p-1)$ and fix $q \ge 2$. If $\{b_j\}$ is any sequence of $I_\alpha(p,q)$-atoms and $\{\lambda_j\}$ is any sequence of scalars such that $\sum |\lambda_j|^p < \infty$ then $f = \sum \lambda_j b_j$ is in $I_\alpha(H^p)$ with $\|f\|_{I_\alpha(H^p)} \le c(\sum |\lambda_j|^p)^{1/p}$. Conversely, if $f \in I_\alpha(H^p)$ then there exist such sequences such that $f = \sum \lambda_j b_j$ and $(\sum |\lambda_j|^p)^{1/p} \le c\|f\|_{I_\alpha(H^p)}$.*

Proof. To show $\sum \lambda_j b_j \in I_\alpha(H^p)$ it suffices to show that $b \in I_\alpha(H^p)$ for any $I_\alpha(p,q)$-atom with $\|b\|_{I_\alpha(H^p)} \le c$ independent of b. By a simple dilation argument we may assume that b is supported in the unit cube. We need to estimate $\|D_{k,\alpha}b\|_p$ for $k > \alpha$. Now we know ([Str]) that $\|D_{k,\alpha}b\|_q \approx \|b\|_{I_\alpha(L^q)}$ so

$$\Big(\int_{|x|\le 2} |D_{k,\alpha}b(x)|^p\,dx\Big)^{1/p} \le c\Big(\int_{|x|\le 2} |D_{k,\alpha}b(x)|^q\,dx\Big)^{1/q}$$
$$\le c\|b\|_{I_\alpha(L^q)} \le c\,.$$

Next suppose $|x| \ge 2$. Then $b(x) = 0$ so

$$\Delta_{ry}^k b(x) = \sum_{m=1}^{k} c_{k,m} b(x + rmy)\,,$$

hence it suffices to estimate

$$\int_{|x|\ge 2} \Big(\int_0^\infty \Big(\int_B |b(x+rmy)|\,dy\Big)^2 \frac{dr}{r^{1+2\alpha}}\Big)^{p/2} dx\,.$$

But $b(x+rmy)=0$ unless $r>|x|/2m$ so this integral is dominated by

$$c\int_{|x|\geq 2}\left(\int_{|x|/2m}^{\infty}(r^{-n}\|b\|_1)^2\frac{dr}{r^{1+2\alpha}}\right)^{p/2}dx\leq c\|b\|_1^p\int_{|x|\geq 2}|x|^{-p(\alpha+n)}\,dx\leq c\|b\|_1^p$$

since $\alpha>n(1/p-1)$, and $\|b\|_1\leq\|b\|_q\leq c\|b\|_{I_\alpha(L^q)}$ since b has support in the unit cube (this last inequality follows from the characterization $\|b\|^q_{I_\alpha(L^q)}\approx\int(\int|\Delta_y^k b(x)|^2dy/|y|^{n+2\alpha})^{q/2}\,dx$ for $q\geq 2$ in [St] and the observation that for $x\in Q$ and $|y|\geq 2$ we have $\Delta_y^k b(x)=\pm b(x)$).

For the converse we take $f\in I_\alpha(H^p)$ so that $f=I_\alpha(g)$ with $g\in H^p$, and take the atomic decomposition of g, $g=\sum\lambda_j a_j$ where a_j are (p,q)-atoms, so $f=\sum\lambda_i I_\alpha(a_j)$ with $(\sum|\lambda_i|^p)^{1/p}\approx\|f\|_{I_\alpha(H^p)}$. However, the $I_\alpha(a_j)$ are not atoms since they do not have compact support. Clearly it suffices to obtain an atomic decomposition for each $I_\alpha(a_j)$, and again by dilation invariance it suffices to do that for $b=I_\alpha(a)$ where a is a (p,q)-atom supported in the unit cube. We then have (2.1) and (2.2) holding for b.

We choose a C^∞ partition of unity $1=\varphi_0+\sum_{m=1}^\infty\varphi_m$ where $\varphi_0\equiv 1$ and $\varphi_1\equiv 0$ on $|x|\leq 2$, $\operatorname{supp}\varphi_0\subseteq\{|x|\leq 4\}$, $\operatorname{supp}\varphi_1\subseteq\{2\leq|x|\leq 8\}$ and $\varphi_m(x)=\varphi_1(2^{1-m}x)$ for $m\geq 2$. We then write $b=\varphi_0 b+\sum_{m=1}^\infty\varphi_m b$ and we have to show $\varphi_m b=\lambda_m b_m$ for appropriate scalars λ_m where b_m are $I_\alpha(p,q)$-atoms and $\sum_{m=0}^\infty|\lambda_m|^p\leq c$. Obviously each b_m is supported on a cube centered at the origin with $|Q|=2^{n(4+m)}$ and so we must take $\lambda_m=2^{n(4+m)(1/p-1/q)}\|\varphi_m b\|_{I_\alpha(L^q)}$ in order that b_m be an atom. Thus we need to estimate $\|\varphi_m b\|_{I_\alpha(L^q)}$.

For $m=0$ we use the fact that $\|b\|_{I_\alpha(L^q)}=1$ and (2.2) to obtain $\|\varphi_0 b\|_{I_\alpha(L^q)}\leq c$. When $m\geq 1$ we use (2.1) instead. It is easiest to give the argument first when α is an integer, for then

$$\|\varphi_m b\|_{I_\alpha(L^q)}\approx\sum_{\|\beta\|=\alpha}\|(\partial/\partial x)^\beta(\varphi_m b)\|_q\,.$$

But by (2.1) we can estimate

$$\left\|\left(\frac{\partial}{\partial x}\right)^\beta(\varphi_m b)\right\|_q^q\leq c2^{nm}\left\|\left(\frac{\partial}{\partial x}\right)^\beta(\varphi_m b)\right\|_\infty^q\leq c2^{nm}2^{qm(-n-j-1)}$$

so

$$\|\varphi_m b\|_{I_\alpha(L^q)}\leq c2^{m(-n/q-j-1)}\text{ and }\lambda_m\leq c2^{m(-n-j-1+n/p)}\,,$$

which gives

$$\sum|\lambda_m|^p\leq c$$

because the exponent is negative.

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