COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

ON THE NUMBER OF POLYNOMIALS OF A UNIVERSAL ALGEBRA, III

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0. Given an algebra $\mathfrak{A}$, let $p_n(\mathfrak{A})$ denote the number of n-ary non-trivial polynomials depending on all n variables. From Theorem 1 in this paper it follows that if $p_0(\mathfrak{A}) = 0$ and if $\mathfrak{A}$ contains a cyclic operation, then $p_{2n+1}(\mathfrak{A}) \geqslant 1$ for all $n \geqslant 1$.

This result can be applied in the investigation of possible representable sequences $\langle a_0, a_1, \ldots\rangle$, that is, sequences for which there exists an algebra $\mathfrak{A}$ such that $a_n = p_n(\mathfrak{A})$ for $n = 0, 1, 2, \ldots$ Some applications of this theorem are given in Section 2. Some results on representable sequences can be found in [1] and [2].

1. In an algebra $\mathfrak{A}$, let $f(x_1, x_2, \ldots, x_{2n+1})$ be a $(2n+1)$-ary cyclic polynomial, that is, a polynomial satisfying the identity

$$f(x_1, x_2, \ldots, x_{2n+1}) = f(x_2, x_3, \ldots, x_{2n+1}, x_1)$$

for some $n \geqslant 1$. Define three ternary operations f_1, f_2, f_3 by

$$f_1(x, y, z) = f(x, \underbrace{y, y, \ldots, y}_{n \text{ times}}, \underbrace{z, z, \ldots, z}_{n \text{ times}}),$$

$$f_2(x, y, z) = f(\underbrace{x, x, \ldots, x}_{2n-1 \text{ times}}, y, z),$$

$$f_3(x, y, z) = f(\underbrace{x, \ldots, x}_{l \text{ times}}, \underbrace{y, \ldots, y}_{m \text{ times}}, \underbrace{z, \ldots, z}_{m \text{ times}}),$$

where $l = n+1$, $m = n/2$ if n is even, and $l = n$, $m = (n+1)/2$ if n is odd. Define a sequence of unary operations by

$$f^0(x) = x,$$
$$f^{k+1}(x) = f\big(f^k(x), f^k(x), \ldots, f^k(x)\big) \quad \text{for } k = 0, 1, 2, \ldots$$

LEMMA 1. *If the operation* $g(f^k(x_1), x_2, \ldots, x_s)$ *is essentially s-ary* $(s \geqslant 1)$ *for some algebraic operation* g, *then at least one of the operations*

$$h_i(x_1, \ldots, x_{s+2}) = g\big(f_i(f^{k-1}(x_1), f^{k-1}(x_2), f^{k-1}(x_3)), x_4, \ldots, x_{s+2}\big), \quad i=1,2,3,$$

is essentially $(s+2)$*-ary.*

Proof. Since f is cyclic, we have the equations

$$f_i(x, y, x) = f_i(x, x, y), \quad i = 1, 2, 3,$$

and from these we get, for $i = 1, 2, 3$,

$$h_i(x, y, x, x_4, \ldots, x_{s+2}) = h_i(x, x, y, x_4, \ldots, x_{s+2}). \tag{1}$$

Set $x_1 = x_2 = x_3$. By using the assumption on the operation g, we see that each operation h_i depends on all of the variables $x_4, x_5, \ldots, x_{s+2}$, and on at least one of the variables x_1, x_2, x_3. Hence each h_i depends on at least s of its $s+2$ variables. If h_i is essentially $(s+1)$-ary, then we must have

$$h_i(x_1, x_2, \ldots, x_{s+2}) = q_i(x_2, x_3, x_4, \ldots, x_{s+2}) \tag{2}$$

for an essentially $(s+1)$-ary operation q_i, because the only other possibilities $q_i(x_1, x_3, x_4, \ldots, x_{s+2})$ and $q_i(x_1, x_2, x_4, \ldots, x_{s+2})$ are eliminated by (1). Similarly, if h_i is essentially s-ary, then we must have

$$h_i(x_1, x_2, \ldots, x_{s+2}) = q_i'(x_1, x_4, x_5, \ldots, x_{s+2}) \tag{3}$$

for an essentially s-ary operation q_i', because the possibilities $q_i'(x_2, x_4, \ldots, x_{s+2})$ and $q_i'(x_3, x_4, \ldots, x_{s+2})$ are also eliminated by (1). Thus, if we assume that Lemma 1 is false, then for each value of $i = 1, 2, 3$ we get either formula (2) or formula (3).

Since f is cyclic, we have

$$f_1(x_1, x_2, x_2) = f_2(x_2, x_2, x_1)$$

and from this we get

$$h_1(x_1, x_2, x_2, x_4, \ldots, x_{s+2}) = h_2(x_2, x_2, x_1, x_4, \ldots, x_{s+2}). \tag{4}$$

Similarly, we have either

$$f_1(x_2, x_2, x_3) = f_3(x_2, x_3, x_3)$$

if n is even, or else, when n is odd, we get

$$f_1(x_2, x_2, x_3) = f_3(x_3, x_2, x_2).$$

These give us one of the equations

$$h_1(x_2, x_2, x_3, x_4, \ldots, x_{s+2}) = \begin{cases} h_3(x_2, x_3, x_3, x_4, \ldots, x_{s+2}), \text{ or} \\ h_3(x_3, x_2, x_2, x_4, \ldots, x_{s+2}). \end{cases} \tag{5}$$

Now take $i = 1$. If (3) holds, then from (4) we get

$$h_2(x_2, x_2, x_1, x_4, \ldots, x_{s+2}) = q_1'(x_1, x_4, \ldots, x_{s+2})$$

contradicting whichever one of (2) or (3) happens to be true for $i = 2$. Similarly, if (2) holds for $i = 1$, then from (5) we get

$$q_1(x_2, x_3, x_4, \ldots, x_{s+2}) = h_3(x_a, x_b, x_b, x_4, \ldots, x_{s+2})$$

with either $a = 3$ and $b = 2$ or $a = 2$ and $b = 3$. But this contradicts whichever one of (2) or (3) happens to be true for $i = 3$. This final contradiction proves Lemma 1.

THEOREM 1. *In an algebra $\mathfrak{A}$, if there exists a cyclic m-ary operation $f(x_1, x_2, \ldots, x_m)$, $m \geqslant 2$, such that $f^k(x)$ is not constant for any k, then in $\mathfrak{A}$ there exists an essentially $(2j+1)$-ary operation for each $j = 1, 2, \ldots$*

Proof. Note that $f(x_1, \ldots, x_m)$ is essentially m-ary, because it is cyclic and $f^1(x) = f(x, x, \ldots, x)$ is not constant.

If $m = 2n$ is even, then we put $x_1 = x_2 = \ldots = x_n = x$ and $x_{n+1} = \ldots = x_m = y$ to obtain the symmetric binary operation $f(x, x, \ldots, x, y, y, \ldots, y)$. By a theorem of Marczewski [3], the algebra has then an essentially j-ary operation for any $j \geqslant 2$.

If $m = 2n+1$ is odd, fix j. The existence of the claimed $(2j+1)$-ary operation will be shown by j applications of Lemma 1. First take $s = 1$, $g(x_1) = x_1$, $k = j$. Then the assumptions of Lemma 1 are satisfied, and so, for a value of i_1, the operation $f_{i_1}\big(f^{j-1}(x_1), f^{j-1}(x_2), f^{j-1}(x_3)\big)$ is essentially ternary. If $j > 1$, we use Lemma 1 again, with $s = 3$, $g(x_1, x_2, x_3) = f_{i_1}\big(x_1, f^{j-1}(x_2), f^{j-1}(x_3)\big)$, $k = j-1$, to get an $i_2 \in \{1, 2, 3\}$ such that the operation

$$f_{i_1}\Big(f_{i_2}\big(f^{j-2}(x_1), f^{j-2}(x_2), f^{j-2}(x_3)\big), f^{j-1}(x_4), f^{j-1}(x_5)\Big)$$

is essentially 5-ary. If $j > 2$, we continue as indicated: $k = j-2$, etc. After j steps we get an essentially $(2j+1)$-ary operation, namely

$$f_{i_1}\Big(f_{i_2}\Big(\ldots\big(f_{i_j}(x_1, x_2, x_3), f^1(x_4), f^1(x_5)\big), f^2(x_6), \ldots \\ \ldots, f^{j-2}(x_{2j-1})\Big), f^{j-1}(x_{2j}), f^{j-1}(x_{2j+1})\Big).$$

2. In this section we will give some applications of Theorem 1.

THEOREM 2. *If $p_0(\mathfrak{A}) = 0$ and $p_{2j+1}(\mathfrak{A}) = 0$ for some $j \geqslant 1$, then m divides $p_m(\mathfrak{A})$ for any prime number m.*

Proof. If p_m is not divisible by m for some m, then there exists an essentially m-ary polynomial $f(x_1, \ldots, x_m)$ such that the number n of distinct operations obtainable from f by permuting its variables is not divisible by m. Let S be the group of symmetries of the polynomial f, that is, the group of all possible permutations of the variables $x_1, x_2, \ldots, x_m$ which do not change the value of f. Then, if S has order s, $m! = s \cdot n$.

Since m is prime, s is divisible by m. By Cauchy's theorem, the group S contains an element of order m. Since m is prime, this element must be a cycle of length m. Thus a certain permutation of f is a cyclic polynomial. Then, by Theorem 1, $p_{2j+1}(\mathfrak{A}) \geqslant 1$ for all $j \geqslant 1$, which is a contradiction.

By combining this Theorem 2 with Case 4 of Theorem 1 of [1], we get

COROLLARY. *If a sequence* $\langle a_0, a_1, \ldots \rangle$ *satisfies* $a_0 = 0, a_1 > 0$, *and* $a_m = 0$ *for all composite* m, *then the sequence is representable if and only if*, m *divides* a_m *for all* m.

Another application of Theorem 1 uses the following Lemma due to G. Grätzer:

LEMMA 2. *If* $1 \leqslant p_m(\mathfrak{A}) < m$ *for some* m, *then* $\mathfrak{A}$ *contains a cyclic* n*-ary polynomial for some* $n > 1$.

Proof. By assumption, we have at least one essentially m-ary polynomial $f(x_1, \ldots, x_m)$. Cyclic permutations of its variables will give m operations, which cannot all be different, since $p_m < m$. So we must have an identity of the form

$$f(x_1, x_2, \ldots, x_m) = f(x_{k+1}, x_{k+2}, \ldots, x_m, x_1, x_2, \ldots, x_k)$$

for some value of k, $1 \leqslant k < m$. If k and m are relatively prime, then f is cyclic. Otherwise, let d be the greatest common divisor of k and m. Then from the last equation we can derive the identity

$$f(x_1, \ldots, x_m) = f(x_{d+1}, x_{d+2}, \ldots, x_m, x_1, x_2, \ldots, x_d).$$

Put $x_1 = x_2 = \ldots = x_d = y_1, x_{d+1} = \ldots = x_{2d} = y_2$, etc. This gives us a cyclic n-ary operation, with $n = m/d$.

By combining Lemma 2 and Theorem 1, we get

COROLLARY. *If* $p_0(\mathfrak{A}) = 0$ *and* $p_{2n+1}(\mathfrak{A}) = 0$ *for some* $n \geqslant 1$, *then, for any* m, *either* $p_m = 0$ *or* $p_m \geqslant m$.

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Reçu par la Rédaction le 8. 10. 1969