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ON THE INTERSECTION OF SOBOLEV SPACES

BY

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1. Introduction. Let p be a fixed real number, $1 \le p < \infty$, r a nonnegative integer and $\Omega \subset \mathbf{R}^n$ a bounded domain. We denote by W^r the space of distributions $f \in L^p(\Omega)$ with derivatives of order not greater than r in L^p and norm $\|f; W^r\| = (\sum_{|\alpha| \le r} \|D^\alpha f\|_p^p)^{1/p}$ (i.e. $W^r = W^{r,p}(\Omega)$ in [A], $= V_p^r$ in [M], $= W_p^r$ in [TR]).

We denote by W_0^r the closure in W^r of $C_0^\infty = C_0^\infty(\Omega)$. If we want to make explicit the index p we write $W^{p,r}$ and $W_0^{p,r}$ instead of W^r and W_0^r respectively, and to underline the fact that $p = 2$, we replace sometimes W by H. We denote by $C^\infty(\overline{\Omega})$ the family of functions f in $C^\infty(\Omega)$ such that $D^\alpha f$ is uniformly continuous in Ω for all α, and define

$$D_r(\Omega) := \{f \in C^\infty(\overline{\Omega}) : \ D^\alpha f = 0 \text{ on } \partial\Omega \text{ for } |\alpha| < r\}.$$

Assume that r and R are positive integers. For $r \le R$ we define $W_{r,R}(\Omega)$,

$$W_{r,R} := W_0^r \cap W^R \quad \text{with norm} \quad \|\cdot; W^R\|.$$

$D_r(\Omega)$ coincides for a large class of regions with $C^\infty(\overline{\Omega}) \cap W_{r,R}(\Omega)$ (cf. end of §2). In [B] we give a constructive proof of the following result.

THEOREM 1. *If Ω has a C^∞ boundary then D_r is dense in $W_{r,R}$.*

In §2 of this paper we exhibit another way of defining the spaces $W_{r,R}$ and in §3 we show that in Th. 1 the requirement $\partial\Omega \in C^\infty$ cannot be replaced by $\partial\Omega \in C^R$.

2. A characterization. Let Ω be as in Th. 1 and $1 < p < \infty$. Prof. A. P. Calderón pointed out to us that $W_{r,R}$ is the (closed) subspace of W^R that consists of the distributions $f \in W^R$ such that $D^\alpha f = 0$ a.e. on $\partial\Omega$ for $|\alpha| < r$. This result can be proved using [C], especially Th. 11. Next we discuss certain details that make more clear this statement and we prove it for a strong Lipschitz domain and $1 \le p < \infty$.

In what follows G will denote a (bounded) *strong Lipschitz domain* (i.e. a domain with the strong local Lipschitz property ([A]) = domain of class $C^{0,1}$ ([M]) = domain with a minimally smooth boundary ([ST])). These domains are Lipschitz domains in the sense given in [M] (also called regions of type K in [MY]). However, both classes do not coincide as can be shown by an example due to Maz'ja ([M], p. 19).

To define a strong Lipschitz domain we follow [G]. Each $x^0 \in \partial G$ has a neighborhood U such that in some orthogonal coordinate system: $x^0 = 0$, $U = \{x : |x_i| < d_i\}$, $U \cap G = \{x : |x_i| < d_i, i = 1, \dots, n-1, \gamma(x') < x_n < d_n\}$, $U \cap \partial G = \{x : |x_i| < d_i, i = 1, \dots, n-1, x_n = \gamma_n(x')\}$, where $x' = (x_1, \dots, x_{n-1})$ and γ is a function such that $|\gamma(x') - \gamma(y')| \le K|x' - y'|$, K a constant. If $d_1, \dots, d_{n-1}$ are sufficiently small then $|\gamma(x')| < d_n/2$ and the map

$$(1) \qquad X_j = x_j/d_j, \qquad X_n = 2(x_n - \gamma(x'))/d_n$$

satisfies a uniform Lipschitz condition as well as its inverse. These functions map the neighborhood of x^0, $R_0 = \{|x_i| < d_i, i < n; \gamma(x') - d_n/2 < x_n < \gamma(x') + d_n/2\}$, onto the cube $R = \{|X_i| < 1, i = 1, \dots, n\}$ and in such a way that the images of $G \cap R_0$ and $\partial G \cap R_0$ are respectively $Q = \{|X_i| < 1, i < n; 0 < X_n < 1\}$, $S = \{|X_i| < 1, i < n; X_n = 0\}$.

Therefore, there exists a finite open covering of ∂G, $\{R_j\}$, $j = 1, \dots, N$, and bilipschitzian transformations Φ_j of the domains R_j onto R such that $R_j \cap G$ and $R_j \cap \partial G$ are transformed by Φ_j onto Q and S respectively.

By definition, a set $A \subset \partial G$ has *surface measure zero* if for each j, $\Phi_j(A \cap R_j)$ has measure zero in the $(n-1)$-dimensional surface S.

To give a sense to the restriction of $u \in W^{1,p}(G)$ to ∂G, that is, to the trace of u, it is sufficient to define the restriction to S of a function in $W^{1,p}(Q)$, since one can use for the general case a partition of unity and the transformations Φ_j. Now, if $f \in W^{1,p}(Q)$ then $\partial f/\partial x_n \in L^p(Q)$ and because of Th. V, p. 57 of [S], there exists in its equivalence class a function absolutely continuous on each segment $\{(x', t) : 0 < t < 1\} \subset Q$. Let us call it a *prototype*. For a prototype $\frac{d}{dt} f(x', t) = \frac{\partial f}{\partial x_n}(x', t) \in L^1(0, 1)$ for a.e. $x' \in S$. In consequence, for a.e. $x' \in S$ the limit $\lim_{t \to 0} f(x', t) =:$ $\operatorname{tr} f(x') =: f(x', 0)$ exists. The definition of $\operatorname{tr} f$, almost everywhere on S, is independent of the prototype chosen. (In regard to this point see also Th. 7.3 of [MY].)

In [G], footnote 7), the following result is proved.

LEMMA 1. *If* $1 \le p < \infty$ *and* $f \in W^{1,p}(Q)$ *then* $\operatorname{tr} f \in L^p(S)$ *and*

$$(2) \qquad \|\operatorname{tr} f; L^p(S)\| \le C_p \|f; W^{1,p}(Q)\|.$$

Another way of defining the trace is as follows. If $u \in C^\infty(\overline{Q})$, $\operatorname{Tr} u(x', 0) := u(x', 0) = \operatorname{tr} u$; if $u \in W^{1,p}(Q)$ and $\{u_m\} \subset C^\infty(\overline{Q})$ is such that $\|u_m - u; W^{1,p}(Q)\| \to 0$ as $m \to \infty$, define $\operatorname{Tr} u := \lim_{m \to \infty} \operatorname{Tr} u_m$. From inequality (2) it follows that this limit exists in $L^p(S)$ and is equal to $\operatorname{tr} u$.

THEOREM 2. *Let* $f \in W^R(G)$, $1 \leq p < \infty$, $1 \leq r \leq R$. *Then* $f \in W_{r,R}$ *if and only if* $\operatorname{tr} D^\alpha f = 0$ *for* $|\alpha| < r$.

To prove this characterization we shall use the following two lemmas.

LEMMA 2. *Let* $f \in W^1(Q)$, $(\operatorname{supp} f)^- \subset R$. *Then*

(i) $f \in W_0^1(Q)$,
(ii) $\operatorname{tr} f = 0$,
(iii) $f^\sim \in W^1(\mathbf{R}^n)$

are equivalent statements. Here $f^\sim$ *is the extension by* 0 *of* f.

Proof. From what we said above it follows that (i) implies (ii). Assume $\operatorname{tr} f = 0$. There is a prototype $F(x', t)$ of f in Q such that its extension by 0 to $\mathbf{R}^n$ is continuous in $-\infty < t < \infty$ for each $x' \in \mathbf{R}^{n-1}$. Therefore $\frac{\partial F}{\partial x_n}(x', t) \in L^p(\mathbf{R}^n)$. By the same Th. V of [S], we get $\partial f^\sim/\partial x_n = \partial F/\partial x_n = (\partial f/\partial x_n)^\sim$ a.e. Analogously $D^\alpha f^\sim = (D^\alpha f)^\sim$ for $|\alpha| = 1$, and $f^\sim \in W^1(\mathbf{R}^n)$.

Finally, if (iii) holds then for $0 < \varepsilon$ sufficiently small $f^\sim(x', x_n - \varepsilon)|_Q \in W_0^1(Q)$. Letting ε tend to 0 we get $\lim f^\sim(x', x_n - \varepsilon) = f$ in $W^1(Q)$, and (i) follows. ∎

LEMMA 3. *Assume that* $r \geq 1$. *If* $D^\alpha f \in W_0^1(G)$ *for all* α *such that* $|\alpha| < r$ *then* $f \in W_0^r(G)$, *and conversely.*

Proof. Since $f \in W_0^1(G)$, $f^\sim \in W^1(\mathbf{R}^n)$ and also $D^\beta f^\sim = (D^\beta f)^\sim$ for $|\beta| = 1$. Likewise, for $|\alpha| < r$ and $|\beta| = 1$, $D^\alpha f \in W_0^1(G)$ implies $(D^\alpha f)^\sim \in W^1(\mathbf{R}^n)$ and $D^\beta (D^\alpha f)^\sim = (D^{\alpha+\beta} f)^\sim$. We obtain by induction for $|\alpha| \leq r$, $D^\alpha f^\sim = (D^\alpha f)^\sim$, and therefore $f^\sim \in W^r(\mathbf{R}^n)$. Given $\varepsilon > 0$, since G is a strong Lipschitz domain, we find by localization a distribution $g \in W^r(\mathbf{R}^n)$ such that $\operatorname{supp} g \Subset G$ and $\|f - g; W^r(G)\| < \varepsilon$. In consequence, $f \in W_0^r(G)$. ∎

Proof of Theorem 2. It is sufficient to prove the theorem for a function f such that the closure of $\operatorname{supp} f$ is contained in some of the open sets R_j. Applying Lemma 3 we find that $f \in W_0^r(R_j \cap G)$ if and only if $D^\alpha f \in W_0^1(R_j \cap G)$, $|\alpha| < r$, which is equivalent to $(D^\alpha f) \circ \Phi_j^{-1} \in W_0^1(Q)$, $|\alpha| < r$. This holds, because of Lemma 2, if and only if $\operatorname{tr} (D^\alpha f \circ \Phi_j^{-1}) = 0$ for $|\alpha| < r$; that is, if and only if $\operatorname{tr} D^\alpha f = 0$ for $|\alpha| < r$. ∎

COROLLARY 1. *Let* $u \in C^{r-1}(\overline{G}) \cap W^R(G)$, $1 \leq p < \infty$, $1 \leq r \leq R$, G *a strong Lipschitz domain. Then* $u \in W_0^r$ *if and only if* $D^\alpha u = 0$ *on* ∂G *for* $|\alpha| < r$.

The useful particular case $r = R = 1$ appears, for example, in [T], Prop. 22.2, and for Lipschitz domains in [MY], Th. 7.1. It follows at once from Corollary 1 that

$$W_{r,R}(G) \cap C^\infty(\overline{G}) = D_r(G). \tag{3}$$

This equality cannot hold without restrictions on G. In fact, let B be the unit open ball in $\mathbf{R}^2$ and $A = B \setminus \{0\}$. Assume that $g \in C^\infty(\overline{B})$, $g(0) \neq 0$, $g = 0$ on $\partial\overline{A} = \partial\overline{B}$, and that ε is a positive sufficiently small number. Define the function $g_\varepsilon(x)$ as 0 on $|x| < \varepsilon$, $= g(x)\log(|x|/\varepsilon)/\log(\delta/\varepsilon)$ if $\varepsilon \leq |x| \leq \delta = \sqrt{\varepsilon}$, $= g$ on $|x| > \delta$. Then $g_\varepsilon \in H_0^1(A)$. Besides

$$\|g_\varepsilon - g; H_0^1(A)\| \to 0 \quad \text{as } \varepsilon \to 0.$$

That is, $g \in C^\infty(\overline{A}) \cap H_0^1(A) \setminus D_1(A)$.

Observe that from the hypothesis on G we have

$$C^\infty(\overline{G}) = \{f|_{\overline{G}} : f \in C^\infty(\mathbf{R}^n)\} \tag{4}$$

because in such regions Whitney's extension theorem can be applied (cf. [W]). (4) is not true for general domains, for example, in $G = B \setminus S$ where S is a sharp Lebesgue spine in $\mathbf{R}^2$ with vertex at 0.

3. Domains with annihilating boundary. We say that a strong Lipschitz domain G has an *annihilating boundary* (or is an *ab-domain*) if for any function $f \in C^\infty(\overline{G})$, $f = 0$ on ∂G implies $D^\alpha f = 0$ on ∂G for all α. We say that G has an *annihilating boundary at* $y \in \partial G$ if there exists an open neighborhood U of y such that for any function $f \in C^\infty(\overline{G})$ with supp $f \subset U$, $f = 0$ on ∂G implies $D^\alpha f = 0$ on ∂G for all α. Briefly, G is an ab-domain iff it has an annihilating boundary at each point of ∂G. Such domains have in fact nowhere smooth boundary as is shown by the next lemma.

LEMMA 4. *G is an ab-domain iff ∂G is nowhere a C^∞-surface.*

Proof. If for some open set U, $G \cap U = \{x \in U : x_n > \phi(x')\}$ $\phi \in C^\infty(\mathbf{R}^{n-1})$, then the function $f(x) = (x_n - \phi(x'))g(x)$, $g \in C_0^\infty(U)$, satisfies $f = 0$, $\partial f/\partial x_n = g$ on ∂G, and therefore G is not an ab-domain. Let $\theta \in C^\infty(\overline{G})$. By (4) it is the restriction to G of a function $F \in C^\infty(\mathbf{R}^n)$. If $F = 0$ on ∂G and $\partial G \cap U$ is nowhere smooth then by the implicit function theorem we must have $\nabla F = 0$ on ∂G. It follows by induction that G is an ab-domain. ■

LEMMA 5. *Suppose that the strong Lipschitz domain G has an annihilating boundary on $\partial G \cap U$ where U is an open neighborhood of a point $y \in \partial G$. Let $\phi \in C_0^\infty(U)$. Then*

(i) *$u \in D_1(G)$ implies $\phi u \in D_r(G)$ for all r,*

(ii) *if $D_r(G)$ is dense in $W_{r,R}(G)$ then $u \in W_{r,R}(G)$ implies $\phi u \in W_0^R(G)$.*

Proof. (i) From the hypothesis, as in Lemma 4, we obtain $\nabla^r(\phi u) = 0$ on ∂G.

(ii) If $\{u_n\} \subset D_r(G)$ is such that $\|u_n - u; W^R(G)\| \to 0$ as $n \to \infty$ then $\|\phi(u_n - u); W^R(G)\| \to 0$. Since, by (i), $\phi u_n \in D_R(G)$, by Corollary 1 we have $\phi u_n \in W_0^R(G)$. ■

We can now prove the main result of this section.

THEOREM 3. *Assume that* $1 \leq r < R$. *If* G *is a strong Lipschitz domain with a* C^R*-boundary which is an ab-domain at a point* $y \in \partial G$ *then* $D_r(G)$ *is not dense in* $W_{r,R}(G)$.

Proof. Let U be the neighborhood of y where ∂G is annihilating. There exist $u \in W_{r,R}(G)$ and $\eta \in C_0^\infty(U)$ such that $\eta u \notin W_0^R(G)$. In fact, we can choose U small enough to have $G \cap U = \{x \in U : x_n > \phi(x')\}$, $\phi \in C^R$. Let $\eta \in C_0^\infty(U)$ such that $\eta(y) = 1$. We define $u(x) = (x_n - \phi(x'))^r \eta(x)$. Then $u \in C^R(\overline{G}) \cap W_0^r(G)$ and $(\partial^r u/\partial x_n^r)(y) = r!$. Because of Corollary 1, $u \notin W_0^{r+1}(G) \supset W_0^R(G)$. Theorem 3 follows now from Lemma 5 (ii). ■

It is easy to construct C^{k+1}-domains, k a nonnegative integer, that are ab-domains: for example, let $w(x)$ be the van der Waerden function and define $G \subset \mathbf{R}^2$ by $G = \{(x,y) : y > \int_0^x (x-t)^k w(t)\, dt\}$.

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