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ORTHOGONALITY AND ADDITIVE FUNCTIONS ON NORMED LINEAR SPACES

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In the paper* we prove the following

THEOREM. *Let V be a real vector space* ($\dim_R V \geqslant 3$) *equipped with a norm* $\| \ \|$ *which does not come from an inner product. A function f: $V \to R$ which is additive on orthogonal elements of V (in the sense of James) is additive on V.*

The above theorem answers a question of Dhombres ([1], 4.79) in the case where $\dim_R V \geqslant 3$, and generalizes a result of Sundaresan [6] in the same situation. Related results have appeared in [2] and [5].

Definition. If V is a normed real linear space, we say that the vector u is *orthogonal* to the vector v, and write $u \perp v$, if $\|u+av\| \geqslant \|u\|$ for all real numbers a.

If u is not orthogonal to v, we write $u \tilde{\perp} v$. If $u \perp v$, then $au \perp bv$ for all real numbers a and b.

A function f: $V \to R$ is *orthogonally additive* if $f(u+v) = f(u)+f(v)$ whenever $u \perp v$.

The concept of orthogonality has been thoroughly investigated by R. James. Among his results we use the following:

1. *If P is a plane (through the origin) in V and u is a vector in P, then there exist nonzero vectors v and w in P such that $u \perp v$ and $w \perp u$* ([3], Corollary 2.2 and Theorem 2.3).

2. *If u and v are nonzero vectors in V and $u \perp v$, then there is a hyperplane P in V such that v is in P and, for every w in P, $u \perp w$* ([3], Theorem 2.1).

3. *If* $\dim_R V \geqslant 3$ *and $u \perp v$ implies $v \perp u$ for all u and v in V (orthogonality is symmetric), then the norm on V comes from an inner product* ([4], Theorem 1).

Proof of the Theorem. It is sufficient to prove that if orthogonality is not symmetric on V, then f is additive. We first consider the case where $\dim_R V = 3$.

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Suppose that $u \perp v$ and $v \tilde{\perp} u$ for some vectors u and v. There exists a nonzero vector w on the plane spanned by u and v such that $w \perp u$. Clearly, w is not a scalar multiple of v; hence, w and v span the plane spanned by u and v. Thus there exist scalars c and d such that $u = cw - dv$. Let a and b be arbitrary real numbers. Then

$$\begin{aligned} f(au)+f(bu)+f(bdv) &= f(au)+f(bu+bdv) = f(au)+f(bcw) = f(au+bcw) \\ &= f([a+b]u+bdv) = f([a+b]u)+f(bdv). \end{aligned}$$

Therefore $f(au)+f(bu) = f([a+b]u)$. We have proved that f is additive on the space spanned by u. Choosing a nonzero vector w' such that $v \perp w'$, we can also show that f is additive on the space spanned by v.

Since $u \perp v$, there exists a plane P (through the origin) such that v is in P and $u \perp w$ for every vector w in P. As $v \tilde{\perp} u$, there exist a real number a and a positive real number ε such that

$$\|v\| > \|v+au\| + \varepsilon.$$

Let the vector w be chosen in P so that w is not a scalar multiple of v and $\|v-w\| < \varepsilon/2$ (since the norm is continuous, this can certainly be done). Then

$$\|v\| < \|w\| + \varepsilon/2 \qquad \text{and} \qquad \|v+au\| > \|w+au\| - \varepsilon/2.$$

Therefore $\|w\| > \|w+au\|$ and, by definition, $w \tilde{\perp} u$. By our earlier result, f is additive on the space spanned by w.

Choose a nonzero z in P such that $v \perp z$. There exist real numbers c and d such that $z = cw - dv$. If a and b are arbitrary real numbers, then

$$\begin{aligned} f(az)+f(bz)+f(adv)+f(bdv) &= f(acw)+f(bcw) = f([ac+bc]w) \\ &= f([a+b]z)+f([ad+bd]v) = f([a+b]z)+f(adv)+f(bdv). \end{aligned}$$

We conclude that $f(az)+f(bz) = f([a+b]z)$; hence, f is additive on the space spanned by z.

Since f is additive on the subspaces spanned by u, v and z, $u \perp P$ and $v \perp z$, f is additive on V.

We now consider the general case. Making use of the Parallelogram Law we can see that there is a two-dimensional subspace generated by u and v on which the norm does not come from an inner product. Let w be an arbitrary vector in V and let V' be a subspace of V containing u, v and w such that $\dim_R V' = 3$. Then f is additive on V', hence on the span of w, and therefore on V.

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