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HARDY-LORENTZ SPACES AND EXPANSIONS IN EIGENFUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR ON COMPACT MANIFOLDS

BY

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Hardy spaces on manifolds have been studied either by considering manifolds as homogeneous spaces in the sense of Coifman–Weiss [C–W] or by appealing to the theory of local Hardy spaces ([Str], [Pee], [Gol]). In the first part of this paper we state several different definitions of Hardy spaces on a compact riemannian manifold M and, besides the spaces $H^p(M)$, we study the more general Hardy–Lorentz spaces $H^{p,q}(M)$ defined through Lorentz norms. Some of the properties of these Hardy–Lorentz spaces have been previously obtained, e.g., in [F–R–S], [Alx], [Col], [Fe–So]. In particular, we state an atomic decomposition for these spaces and some duality results (which have been obtained in cooperation with Björn Jawerth of Washington University in St. Louis). Some of these results and proofs are perhaps new even in the euclidean case. In the second part of the paper we extend some classical inequalities of Paley, Hardy, and Littlewood to expansions in eigenfunctions of the Laplace–Beltrami operator of M. The case where M is a compact Lie group or a symmetric space is also considered.

1. Preliminaries. Let M be a connected compact C^∞-manifold of dimension N endowed with smooth riemannian metric d and riemannian measure dx. As usual, $|A|$ denotes the measure of a measurable subset A of M. $L^p(M)$, $0 < p \leqslant \infty$, is the Lebesgue space on M, while $L^{p,q}(M)$, $0 < p < \infty$, $0 < q \leqslant \infty$, is the Lorentz space defined by the (quasi) norm

$$\|f\|_{L^{p,q}} = \left(\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q},$$

where f^* is the nonincreasing rearrangement of f (see [B–L]).

$C^k(M)$, $k \geqslant 0$, is the space of k-times differentiable functions on M. Let $\{U_j\}$ be a finite open covering of M, and $\{\psi_j\}$ be diffeomorphisms of the U_j's onto the unit ball in $\boldsymbol{R}^N$. Finally, let $\{\varphi_j\}$ be a partition of unity on M associated with the covering $\{U_j\}$, and assume that the support of each φ_j is a compact subset of U_j. We can norm $C^k(M)$ by

$$\|f\|_{C^k} = \sum_j \|(\varphi_j f)\circ\psi_j^{-1}\|_{C^k(\boldsymbol{R}^N)}.$$

$\mathcal{S}(M)$ is the Schwartz space of indefinitely differentiable functions on M, and $\mathcal{S}'(M)$ is its dual, the space of distributions. $\langle f, \phi\rangle$ denotes the pairing between a distribution f and a smooth function ϕ.

We denote by Δ the Laplace–Beltrami operator of M. The solution of the boundary value problem on $\boldsymbol{R}_+ \times M$

$$\Delta u(t, x) = (\partial/\partial t)\, u(t, x), \qquad u(0, x) = f(x)$$

is given by the "convolution" with the heat kernel $W(t, x, y)$ defined on $\boldsymbol{R}_+ \times M \times M$,

$$u(t, x) = \langle f, W(t, x, \cdot)\rangle.$$

Let us denote by $0 = -\lambda_0 \geqslant -\lambda_1 \geqslant -\lambda_2 \geqslant \ldots$ the eigenvalues of Δ, and by $\{\phi_j\}$ the associated orthonormal complete system of eigenvectors. Then

$$W(t, x, y) = \sum_{j=0}^{\infty} \exp(-\lambda_j t)\, \phi_j(x)\, \phi_j(y).$$

It is also possible to give an asymptotic expansion of the heat kernel. In particular,

$$W(t, x, y) = t^{-N/2} \exp\big(-d^2(x, y)/4t\big)\, G(t, x, y),$$

where the function G satisfies $\|G(t, x, \cdot)\|_{C^k} \leqslant c_k$ for every $k \geqslant 0$, with c_k independent of $t \in \boldsymbol{R}_+$ and $x \in M$ (see [B–G–M] or [Cha]).

2. Hardy spaces. Let f be a distribution on M and let

$$u(t, x) = \langle f, W(t, x, \cdot)\rangle.$$

(i) The *radial maximal function* $W^+ f$ is the function

$$W^+ f(x) = \sup\{|u(t, x)|:\ t > 0\}.$$

(ii) The *nontangential maximal function* $W_\alpha^+ f$, $\alpha > 0$, is the function

$$W_\alpha^+ f(x) = \sup\{|u(t, z)|:\ t > 0,\ z \in M,\ d(z, x) < \alpha\sqrt{t}\}.$$

(iii) The *tangential maximal function* $W_\beta^{++}(f)$, $\beta > 0$, is the function

$$W_\beta^{++} f(x) = \sup\left\{|u(t, z)| \left(1 + \frac{d(z, x)}{\sqrt{t}}\right)^{-\beta} :\ t > 0,\ z \in M\right\}.$$

Let s be a nonnegative integer and let x be a point of M. $K_s(x)$ is the set of all ϕ in $\mathcal{S}(M)$ for which

(a) ϕ is supported by a ball B centered at x;

(b) $\|\phi\|_{C^k} \leqslant |B|^{-1-k/N}$, $\quad k = 0, 1, \ldots, s$.

(iv) The *grand maximal function* f_s^* is the function

$$f_s^*(x) = \sup\{|\langle f, \phi\rangle|:\ \phi \in K_s(x)\}.$$

2.1. Proposition. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $\alpha > 0$, $\beta > N/p$, *and* $s > N/p$. *Then the* $L^{p,q}$*-norms of the maximal functions* (i)–(iv) *are equivalent.*

The proof of this proposition is by now quite standard, and it is similar to the corresponding proofs in [Fe–St] and [Fo–St] for the case of L^p-norms on euclidean spaces and homogeneous groups. One just needs the expression of the heat kernel given in the preceding section. Working on a compact manifold and dealing with Lorentz norms create no serious extra difficulties.

Definition. The *Hardy space* $H^{p,q}(M)$, $0 < p < \infty$, $0 < q \leqslant \infty$, is the set of all distributions f on M with grand maximal function $f^*_{[N/p]+1}$ in the Lorentz space $L^{p,q}(M)$. The $H^{p,q}$*-norm* of f is

$$\|f\|_{H^{p,q}} = \|f^*_{[N/p]+1}\|_{L^{p,q}}.$$

If $q = p$, we just have the classical Hardy space $H^p(M)$. This is the definition of Hardy space on a manifold given by Peetre in [Pee], however the previous proposition shows that any of the maximal functions (i)–(iv) can be used equally well to define $H^{p,q}(M)$.

The structure of manifold relates naturally the above definition of Hardy space with the notion of local Hardy space on $\boldsymbol{R}^N$. Let us define the local Hardy space $h^{p,q}(\boldsymbol{R}^N)$ as in [Gol]. Then one easily sees that the following proposition holds:

2.2. Proposition. *Let* $\{U_j, \psi_j\}$ *be coordinate charts and let* $\{\varphi_j\}$ *be a partition of unity as in Section* 1. *Then a distribution f is in the Hardy space* $H^{p,q}(M)$ *if and only if, for every j,* $(f\varphi_j)\circ\psi_j$ *is in the local Hardy space* $h^{p,q}(\boldsymbol{R}^N)$.

As a consequence of this proposition one can lift results from $h^{p,q}(\boldsymbol{R}^N)$ to $H^{p,q}(M)$. For example:

2.3. Proposition. *Let* $0 < p \leqslant r < \infty$, $0 < q \leqslant \infty$, *and let*

$$\alpha = N(1/p - 1/r).$$

Then if T is a pseudodifferential operator on M in the symbol class $S^{-\alpha}_{1,0}$, *T maps* $H^{p,q}(M)$ *into* $H^{r,q}(M)$ *continuously.*

If $p = q = r$, and hence $\alpha = 0$, then this proposition is contained in [Str], [Pee], [Gol]. In our case observe that a pseudodifferential operator in the symbol class $S^{-\alpha}_{1,0}$ can be factorized into a Bessel potential (or fractional integral operator) of index α, which maps $h^p(\boldsymbol{R}^N)$ into $h^r(\boldsymbol{R}^N)$, and a pseudodifferential operator in the symbol class $S^0_{1,0}$, which preserves $h^r(\boldsymbol{R}^N)$. The corresponding result for Hardy–Lorentz spaces can be easily achieved through interpolation.

3. Atomic decomposition. In this section we want to introduce an atomic decomposition for $H^{p,q}(M)$, analogous to the classical atomic decomposition of $H^p(\boldsymbol{R}^N)$. Atoms on $\boldsymbol{R}^N$ are bounded functions with compact support which satisfy certain moment conditions. Therefore we have to introduce a class of functions on M that play the role of the polynomials. A natural choice, but not the only one, is the following:

Let z be a point of M, let T_z ($\cong R^N$) be the tangent space at z, and let Exp_z be the exponential map at z. Then there exists an $r_0 > 0$ such that, for every z in M, Exp_z is injective from the ball $B(0, r_0)$ in T_z into the ball $B(z, r_0)$ in M. The largest r_0 with this property is called the *injectivity radius* of M, and in the sequel any ball in M will be assumed to have radius smaller than this injectivity radius.

Given a ball $B(z, r)$ in M, we define the *polynomials* on $B(z, r)$ as the images of the polynomials on the tangent space T_z via the exponential map, i.e.,

$$P(\mathrm{Exp}_z X) = \sum_j c_j X^j.$$

Let $0 < p \leqslant 1$. A *regular (p, ∞, s)-atom* is a function a in $L^\infty(M)$ satisfying:
(a) the support of a is contained in a ball B;
(b) $\|a\|_{L^\infty} \leqslant |B|^{-1/p}$;
(c) $\int_M a(x)P(x)\,dx = 0$ for every polynomial P on B of degree at most s.

An *exceptional atom* is any function a in $L^\infty(M)$ satisfying

$$\|a\|_{L^\infty} \leqslant 1.$$

3.1. PROPOSITION. *Let f be a distribution in $H^{p,q}(M)$, $0 < p \leqslant 1$, $0 < q \leqslant \infty$, and*

$$\Omega_k = \{x \in M\colon f^*_{[N/p]+1}(x) > (f^*_{[N/p]+1})^*(2^k)\} \quad \textit{for } k \in Z$$

(so that $|\Omega_k| = 2^k$). Then, if s is a nonnegative integer, there exists a sequence of bounded functions $\{f_k\}$ with the following properties:

(i) $$\operatorname{supp} f_k \subseteq \Omega_k;$$

(ii) $$f_k(x) = c(f^*_{[N/p]+1})^*(2^{k-1}) \sum_j |B_{kj}|^{1/p} a_{kj}(x),$$

where the a_{kj}'s are (p, ∞, s)-atoms with supports in the balls B_{kj}, and these balls are a covering of Ω_k with uniformly bounded overlaps;

(iii) $$f = \lim_{n\to+\infty} \sum_{|k|\leqslant n} f_k$$

in the topology of $H^{p_0}(M) + H^{p_1}(M)$ if

$$0 < p_0 < p < p_1 \leqslant \infty \quad \textit{and} \quad s \geqslant [N(1/p_0 - 1)].$$

The proof of this proposition relies on the Calderón–Zygmund decomposition of the distribution f and does not differ essentially from the classical proof of the atomic decomposition of $H^p(\mathbf{R}^N)$. (Again see [Fo–St] as a comprehensive reference or [Fe–So] for a proof of the atomic decomposition of $H^{1,\infty}(\mathbf{R}^N)$.)

Before we state the converse of this proposition observe that for the sequence $\{f_k\}$ we have

$$\|f_k\|_{H^{p_i}} \leqslant c\cdot 2^{k/p_i}(f^*_{[N/p]+1})^*(2^{k-1}), \quad i = 0, 1,$$

and, by the definition of the $H^{p,q}$-norm, the sequence

$$\{2^{k/p}(f^*_{[N/p]+1})^*(2^k)\}$$

is in the sequence space l^q.

3.2. Proposition. *Let $0 < p_0 < p < p_1 \leqslant \infty$, $0 < q \leqslant \infty$, and let $\{f_k\}$ be a sequence of distributions with*

$$\|f_k\|_{H^{p_i}} \leqslant 2^{k/p_i}\alpha_k, \quad i = 0, 1.$$

Then, if the sequence $\{2^{k/p}\alpha_k\}$ is in l^q, then $\sum_k f_k$ converges to a distribution in $H^{p,q}(M)$, and

$$\|\sum_k f_k\|_{H^{p,q}} \leqslant c(\sum_k |2^{k/p}\alpha_k|^q)^{1/q}.$$

To understand this theorem we use the functional J of the real method of interpolation (see [B–L]). Set

$$a = 2^{1/p_0-1/p_1} \quad \text{and} \quad (1-\theta)/p_0+\theta/p_1 = 1/p.$$

One easily sees that

$$J(a^k, f_k) = \max\{\|f_k\|_{H^{p_0}}, a^k\|f_k\|_{H^{p_1}}\} \leqslant a^{\theta k}\cdot 2^{k/p}\alpha_k.$$

Thus $\{a^{-\theta k}J(a^k, f_k)\}$ is in l^q, and $\sum_k f_k$ is in the interpolation space $(H^{p_0}(M), H^{p_1}(M))_{\theta,q}$. Consider now the grand maximal function operator $f \to f^*_{[N/p_0]+1}$. This operator maps $H^{p_i}(M)$ into $L^{p_i}(M)$, therefore it maps $(H^{p_0}(M), H^{p_1}(M))_{\theta,q}$ into

$$(L^{p_0}(M), L^{p_1}(M))_{\theta,q} = L^{p,q}(M),$$

and the proposition follows.

Finally, observe that from the above propositions we easily obtain another proof of the following interpolation result of Fefferman et al. [F–R–S].

3.3. Proposition. *Let $0 < p_0 < p < p_1 \leqslant \infty$, $0 < q_0, q_1, q \leqslant \infty$, and let*

$$(1-\theta)/p_0+\theta/p_1 = 1/p.$$

Then

$$(H^{p_0,q_0}(M),\ H^{p_1,q_1}(M))_{\theta,q} = H^{p,q}(M).$$

Remark. When M is a compact Lie group, it may be more natural to consider atoms with moments with respect to the trigonometric polynomials of the group, so that these moments are preserved under convolution operators. For these atoms it is possible to prove an analogue of Proposition 3.1. Some parts of the proof differ from the classical arguments, however we prefer to omit the proof for the sake of brevity.

4. Duality. Let ϕ be an integrable function on M and let s be a non-negative integer. Then for every ball B in M there exists a unique polynomial ϕ_B on B of degree at most s, such that

$$\int_B (\phi(x)-\phi_B(x))\,P(x)\,dx = 0$$

for every polynomial P on B of degree at most s.

If Ω is an open set in M, and if

$$\Omega = \bigcup_j B_j,$$

where $\{B_j\}$ is a covering of Ω with balls with uniformly bounded overlaps, we define the "oscillation" of ϕ on Ω as

$$O_s(\Omega, \phi) = \sup\Big\{\sum_j \int_{B_j} |\phi(x)-\phi_{B_j}(x)|\,dx\colon\ \bigcup_j B_j = \Omega\Big\},$$

where the supremum is taken with respect to all coverings with uniformly bounded overlaps. We also define the "modulus of continuity" as

$$\omega_s(t, \phi) = t^{-1}\sup\{O_s(\Omega, \phi)\colon\ |\Omega| \leqslant t\}.$$

Definition. The *Campanato space* $X^{\alpha,q}(M)$, $\alpha \geqslant 0$, $0 < q \leqslant \infty$, is the set of all integrable functions ϕ on M with the sequence $\{2^{-k\alpha}\omega_{[N\alpha]}(2^k, \phi)\}$ in l^q. The $X^{\alpha,q}$*-norm* of ϕ is

$$\|\phi\|_{X^{\alpha,q}} = \Big(\sum_k |2^{-k\alpha}\omega_{[N\alpha]}(2^k, \phi)|^q\Big)^{1/q}.$$

It is easy to see that

$$\sup\Big\{|B|^{-1-\alpha}\int_B |\phi(x)-\phi_B(x)|\,dx\colon\ B\Big\}$$

is an equivalent norm in $X^{\alpha,\infty}(M)$. In particular, $X^{0,\infty}(M)$ coincides with the space of functions of bounded mean oscillation BMO(M), and, when $\alpha > 0$, $X^{\alpha,\infty}(M)$ coincide with the spaces introduced by Campanato in [Cam]. In general, it is possible to prove that for $\alpha > 0$ the spaces $X^{\alpha,q}(M)$ coincide with the Besov–Lipschitz spaces $B_{\infty}^{N\alpha,q}(M)$ (see [B–L] and [Tri] for a definition of Besov spaces, and [C–J] for the relation with these Campanato spaces).

In the next proposition, with a slight abuse of notation, by $H^{p,\infty}(M)$ we shall mean the closure of the set of smooth functions in the $H^{p,\infty}$-norm.

4.1. PROPOSITION. *Let* $0 < p \leqslant 1$, $\alpha = N(1/p-1)$, $0 < q \leqslant \infty$, *and let* $q' = +\infty$ *if* $0 < q \leqslant 1$ *and* $q' = q/(q-1)$ *if* $1 \leqslant q \leqslant \infty$. *Then the dual space of the Hardy space* $H^{p,q}(M)$ *can be naturally identified with the Campanato space* $X^{\alpha,q'}(M)$.

This proposition follows from the atomic decomposition of the spaces $H^{p,q}(M)$ through the techniques developed in [C–W] and [Fe–So]. Details are in [C–J].

5. Hardy type inequalities for eigenfunction expansions. For the Fourier transform of a distribution f on the N-dimensional torus one has the classical inequality

$$\Big(\sum_j |j|^{N(p-2)} |\hat{f}(j)|^p\Big)^{1/p} \leqslant c\|f\|_{H^p} \qquad (0 < p \leqslant 2),$$

which, in the one-dimensional case, is due to Paley when $1 < p \leqslant 2$, and to Hardy and Littlewood when $0 < p \leqslant 1$. The purpose of this section is a faithful extension of this inequality to expansions in eigenvectors of the Laplace–Beltrami operator of M.

We start by proving an imbedding of Hardy spaces into Besov spaces, which may be of interest by itself.

DEFINITION. The *Besov space* $B_2^{\alpha,q}(M)$, $-\infty < \alpha < +\infty$, $0 < q \leqslant \infty$, is the set of all distributions f on M with

$$\|f\|_{B_2^{\alpha,q}} = \Big\{\sum_k \Big|2^{k\alpha}\Big(\sum_{2^k \leqslant \sqrt{\lambda_j} < 2^{k+1}} |\langle f, \phi_j\rangle|^2\Big)^{1/2}\Big|^q\Big\}^{1/q} < +\infty.$$

Define the "fractional integral" operator I^β, $\beta > 0$, by

$$I^\beta f = \sum_j \lambda_j^{-\beta/2} \langle f, \phi_j\rangle \phi_j.$$

It is known that this operator is a pseudodifferential operator in the symbol class $S_{1,0}^{-\beta}$ (see [See] or [Tay]). Then, if

$$0 < p < r < \infty, \quad 0 < q \leqslant \infty, \quad \text{and} \quad \beta = N(1/p - 1/r),$$

Proposition 2.3 implies that

$$I^\beta\colon H^{p,q}(M) \to H^{r,q}(M).$$

In particular, for $r = q = 2$, we get the inclusion

$$H^{p,2}(M) \subseteq B_2^{N(1/2-1/p),2}(M).$$

This result and an easy interpolation argument imply

5.1. PROPOSITION. *Let* $0 < p < 2$, $0 < q \leqslant \infty$, *and let* $\alpha = N(1/2 - 1/p)$. *Then the Hardy space* $H^{p,q}(M)$ *is imbedded into the Besov space* $B_2^{\alpha,q}(M)$.

The above proposition is the key for a quick proof of a generalized Hardy inequality.

5.2. PROPOSITION. *Let* $0 < p < 2$, $0 < q \leqslant 2$, *and let* $\{m_j\}$ *be a sequence of nonnegative numbers such that*

$$\Big(\sum_{2^k \leqslant \sqrt{\lambda_j} < 2^{k+1}} m_j^{2/(2-q)}\Big)^{(2-q)/2} \leqslant c \cdot 2^{kN(1/2-1/p)q}.$$

Then

$$\Big(\sum_j m_j |\langle f, \phi_j \rangle|^q\Big)^{1/q} \leqslant c \|f\|_{H^{p,q}}.$$

To prove the proposition observe that

$$\sum_{2^k \leqslant \sqrt{\lambda_j} < 2^{k+1}} m_j |\langle f, \phi_j \rangle|^q$$

$$\leqslant 2^{kN(1/p-1/2)q}\Big(\sum_{2^k \leqslant \sqrt{\lambda_j} < 2^{k+1}} m_j^{2/(2-q)}\Big)^{(2-q)/2} \cdot 2^{kN(1/2-1/p)q}\Big(\sum_{2^k \leqslant \sqrt{\lambda_j} < 2^{k+1}} |\langle f, \phi_j \rangle|^2\Big)^{q/2}.$$

Summation with respect to k completes the proof.

5.3. COROLLARY (Hardy inequality). *Let* $0 < p \leqslant 2$. *Then*

$$\Big(\sum_j \lambda_j^{N(p-2)/2} |\langle f, \phi_j \rangle|^p\Big)^{1/p} \leqslant c \|f\|_{H^p}.$$

To prove the corollary it is enough to show that the sequence $\{\lambda_j^{N(p-2)/2}\}$ satisfies the assumption of Proposition 5.2 (for $q = p$). This follows from the Weyl formula (see [Cha])

$$\operatorname{card}\{\lambda_j \colon \lambda_j \leqslant \lambda\} \approx \omega_N (2\pi)^{-N} |M| \lambda^{N/2} \qquad (\lambda \to +\infty).$$

The above result shows a strong analogy with the euclidean case. This is somewhat surprising since our "Fourier transform" $\{\langle f, \phi_j \rangle\}$ behaves quite differently from its euclidean counterpart. As an example, if $2 < q \leqslant \infty$, then the sequence $\{\|\phi_j\|_{L^q}\}$ may be unbounded, so that if f is in $L^p(M)$, $1 \leqslant p < 2$, then the sequence $\{\langle f, \phi_j \rangle\}$ may also be unbounded (see, e.g., [G–T]). In this context we also note that a strict analogue of the Paley lacunary inequality for functions on the torus,

$$\Big(\sum_j |\hat{f}(2^j)|^2\Big)^{1/2} \leqslant c \|f\|_{H^p} \qquad (1 \leqslant p \leqslant 2),$$

does not work for our general eigenfunction expansions. To make the inequality work in this context one should introduce a suitable weight.

Remark. The case where M is a compact Lie group or a symmetric space is of particular interest, and it has been previously considered by several authors (see, e.g., [C–W], [Gia], [C–G]).

Let (G, K) be a compact symmetric pair and let $\{\pi\}$ be the set of irreducible unitary representations of G of class one with respect to K. This set is naturally identified with a cone in Z^R, where R is the rank of G/K, and we shall denote by $|\pi|$ the norm of π. With any distribution f on G/K we associate its Fourier series $\sum_\pi f_\pi$, where f_π denotes the projection of f into the subspace of $L^2(G/K)$ associated with π.

5.4. PROPOSITION. *Let G/K be a compact symmetric space of dimension N and rank R, and let $0 < p \leqslant 2$. Then, if*

$$f = \sum_\pi f_\pi$$

is in $H^p(G/K)$, we have

$$\Big(\sum_\pi |\pi|^{(N+R)(p-2)/2} \|f_\pi\|_{L^2}^p\Big)^{1/p} \leqslant c\|f\|_{H^p}.$$

The case $p = 1$ of this proposition has been proved by using the atomic decomposition of $H^1(G/K)$ by Giacalone [Gia] and Cazzaniga and Giacalone [C–G], but the case of the euclidean sphere was already contained in [C–W]. Our approach to this problem is different, and again relies on Proposition 5.1.

Observe that there are approximately 2^{kR} representations in

$$\{\pi\colon 2^k \leqslant |\pi| < 2^{k+1}\},$$

so that, by the Hölder inequality,

$$\sum_{2^k \leqslant |\pi| < 2^{k+1}} |\pi|^{(N+R)(p-2)/2} \|f_\pi\|_{L^2}^p \leqslant c \cdot 2^{kN(1/2-1/p)p} \Big(\sum_{2^k \leqslant |\pi| < 2^{k+1}} \|f_\pi\|_{L^2}^2\Big)^{p/2}.$$

Adding over k and using Proposition 5.1 we obtain the desired result.

In order to prove that the index $[(N+R)/2](p-2)$ in the above proposition is the best possible consider the derivatives of the heat kernel

$$\frac{d^n}{dt^n} W(t, x, y).$$

Using the estimates in Section 1 one has, if

$$n > \frac{N}{2}(1/p-1) \quad \text{and} \quad t \to 0,$$

then

$$\left\| \frac{d^n}{dt^n} W(t, x, y) \right\|_{H^p} \simeq t^{(N/2)(1/p-1)-n}.$$

Also,

$$\left\| \left(\frac{d^n}{dt^n} W(t, x, y) \right)_\pi \right\|_{L^2} = \lambda_\pi^n \exp(-\lambda_\pi t) \sqrt{d_\pi},$$

where $-\lambda_\pi$ and d_π denote the eigenvalue and the dimension of the representation π, respectively. The formula for the eigenvalues and the Weyl dimension formula ([Wal]) imply that $\lambda_\pi = |\pi|^2 + O(|\pi|)$, and $d_\pi \simeq |\pi|^{N-R}$, at least for "most" π's. We can then conclude that if

$$n > \frac{N}{2}(1/p - 1) \quad \text{and} \quad t \to 0,$$

then

$$\left\{ \sum_\pi |\pi|^{(N+R)(p-2)/2} \left\| \left(\frac{d^n}{dt^n} W(t, x, y) \right)_\pi \right\|_{L^2}^p \right\}^{1/p} \simeq \left\| \frac{d^n}{dt^n} W(t, x, y) \right\|_{H^p}.$$

REFERENCES

[Alx] A. B. Alexandrov, *Essays on non locally convex Hardy classes*, pp. 1–89 in: Lecture Notes in Math. 864, Springer-Verlag, Berlin 1981.

[B–G–M] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Math. 194, Springer-Verlag, Berlin 1974.

[B–L] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin 1976.

[Cam] S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 18 (1964), pp. 137–160.

[C–G] F. Cazzaniga and E. Giacalone, *Hardy's inequality for compact symmetric spaces*, Boll. Un. Mat. Ital. A (6) 2 (1983), pp. 381–388.

[Cha] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando 1984.

[C–W] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), pp. 569–645.

[Col] L. Colzani, *Taylor coefficients of functions in certain weak Hardy spaces*, Boll. Un. Mat. Ital. A (6) 4 (1985), pp. 57–66.

[C–J] — and B. Jawerth, *The duals of certain Hardy spaces.*

[Gia] E. Giacalone, *Hardy's inequality for compact Lie groups*, J. Reine Angew. Math. 338 (1983), pp. 144–148.

[F–R–S] C. Fefferman, N. Rivière and Y. Sagher, *Interpolation between H^p spaces: the real method*, Trans. Amer. Math. Soc. 191 (1974), pp. 75–82.

[Fe–So] R. Fefferman and F. Soria, *The space weak H^1*, Studia Math. 85 (1987), pp. 1–16.

[Fe–St] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), pp. 137–193.

[Fo–St] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, Princeton 1982.

[G–T] S. Giulini and G. Travaglini, *L^p-estimates for matrix coefficients of irreducible representations of compact groups*, Proc. Amer. Math. Soc. 80 (1980), pp. 448–450.

[Gol] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. 46 (1979), pp. 27–42.

[Pee] J. Peetre, *Classes de Hardy sur les variétés*, C. R. Acad. Sci. Paris Sér. I Math. 280 (1975), pp. 439–441.

[See] R. Seeley, *Complex powers of an elliptic operator*, pp. 288–307 in: Proc. Symp. Pure Math. Vol. 10, A.M.S., Providence 1967.

[Str] R. S. Strichartz, *The Hardy space H^1 on manifolds and submanifolds*, Canad. J. Math. 24 (1972), pp. 915–925.

[Tay] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton 1981.

[Tri] H. Triebel, *Spaces of Besov–Hardy–Sobolev type on complete riemannian manifolds*, Ark. Mat. 24 (1986), pp. 299–337.

[Wal] N. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York 1973.

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