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ON DYE'S CONDITION IN NILPOTENT GROUPS OF CLASS 2

BY

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In [1] and [2] Dye developed a theory of approximation of automorphisms of a probability measure space (M, μ) by finite groups of automorphisms. Among other things he proved that any Abelian group of automorphisms of (M, μ) admits such an approximation. He showed also that if a group G satisfies the condition

$$\text{(D)} \qquad \inf_n \frac{|S^{2n} - S^{2n-1}|}{|S^n|} = 0$$

for some finite subset S of G such that $1 \in S$, $S^{-1} = S$ and $\bigcup S^n = G$, then G admits approximation by finite groups.

The aim of this paper * is to verify that nilpotent groups of class 2 on two generators satisfy (D), and thus to obtain some more information about the action of these groups as automorphisms of a probability measure space.

1. Notation. Let G be the free nilpotent group of class 2 on two generators. It can be viewed as the group of words $x^a y^b [y, x]^c$, where a, b, c are integers and multiplication is defined by the formula

$$x^a y^b [y, x]^c \cdot x^{a'} y^{b'} [y, x]^{c'} = x^{a+a'} y^{b+b'} [y, x]^{c+c'+a'b}.$$

Let us denote by F the set $\{1, x, x^{-1}, y, y^{-1}\}$ and let, for a set $A \subset G$,

$$A^n = \{a_1 a_2 \dots a_n\colon\ a_i \in A,\ i = 1, 2, \dots, n\}.$$

Clearly, $F^n \subset F^{n+1}$, and $\bigcup F^n = G$. For a triple k, l, n of integers such that $|k| + |l| \leqslant n$, we write

$$C_n(k, l) = \{c\colon\ x^k y^l [y, x]^c \in F^n\},$$

$$M_n(k, l) = \max C_n(k, l) \quad \text{and} \quad m_n(k, l) = \min C_n(k, l).$$

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The following identities, valid for every nilpotent group of class 2 and an arbitrary integer n, will be used throughout:

$$[u^n, v] = [u, v^n] = [u, v]^n \quad \text{and} \quad (uv)^n = u^n v^n [v, u]^{n(n-1)/2}.$$

2. Lemmas.

LEMMA 1. *For all k, l, n, we have*

$$C_n(k, l) = \{s \in N\colon m_n(k, l) \leqslant s \leqslant M_n(k, l)\}.$$

Proof. If $0 < c \in C_n(k, l)$, then in F^n there is an element of the form $ayxb$ or $ay^{-1}x^{-1}b$, where $a \in F^s$, $b \in F^t$ and $s+t \leqslant n-2$. Thus either $axyb$ or $ax^{-1}y^{-1}b$ belongs to F^n. Hence $c-1 \in C_n(k, l)$. Similarly, if $c < 0$, then $c+1 \in C_n(k, l)$. Since $0 \in C_n(k, l)$, the proof of the lemma is complete.

LEMMA 2. A. *There exist integers a_1, a_2, b_1, b_2 such that $a_1 + a_2 = k$, $b_1 + b_2 = l$ and either $M_n(k, l) = a_2 b_1$ or $M_n(k, l) = a_1 b_2 + a_1 b_1 + a_2 b_2$.*

B. *There exist integers a_1, a_2, b_1, b_2 such that $a_1 + a_2 = k$, $b_1 + b_2 = l$ and either $m_n(k, l) = a_2 b_1$ or $m_n(k, l) = a_1 b_1 + a_1 b_2 + a_2 b_1$.*

Proof. Let

$$w = x^{c_1} y^{d_1} \ldots x^{c_t} y^{d_t} = x^a y^b [y, x]^e,$$

where

$$a = a(w) = \sum_i c_i, \quad b = b(w) = \sum_i d_i, \quad e = e(w) = \sum_{j<i} c_i d_j.$$

A. Let $w = x^{c_1} y^{d_1} x^{c_2} y^{d_2} x^{c_3}$. We are going to show that there is a word w' of length not greater than 4 and such that $a(w') = a(w)$, $b(w') = b(w)$ and $|e(w)| < |e(w')|$. Assume that $c_i \neq 0$ for $i = 1, 2, 3$, and $d_i \neq 0$ for $i = 1, 2$. Consider two cases for $e(w) = c_2 d_1 + c_3 d_1 + c_3 d_2$.

1. $e(w) > 0$.

If $c_2 d_1 < 0$, then we put $w' = x^{c_1+c_2} y^{d_1+d_2} x^{c_3}$. Thus $e(w') = e(w) - c_2 d_1$.

If $c_2 d_1 > 0$ and $c_3 d_2 < 0$, then we put $w' = x^{c_1} y^{d_1} x^{c_2+c_3} y^{d_2}$. Thus $e(w') = e(w) - c_3 d_2$.

If $c_2 d_1 > 0$, $c_3 d_2 > 0$ and $c_3 d_1 > 0$, then $\operatorname{sgn} c_2 = \operatorname{sgn} c_3 = \operatorname{sgn} d_1 = \operatorname{sgn} d_2$, and we put $w' = x^{c_1} y^{d_1+d_2} x^{c_2+c_3}$, which yields $e(w') = e(w) + c_2 d_2$.

If $c_1 d_1 > 0$, then we put $y^{d_1} x^{c_1+c_2} y^{d_2} x^{c_3}$. Hence $e(w') = e(w) + c_1 d_1$.

If $c_1 d_1 < 0$ and $(d_1 + d_2) c_3 < 0$, then we put $w' = x^{c_1+c_3} y^{d_1} x^{c_2} y^{d_2}$. This yields $e(w') = e(w) - (d_1 c_3 + d_2 c_3)$.

If $c_1 d_1 < 0$, $(d_1 + d_2) c_3 > 0$ and $c_3 d_1 < 0$, then $\operatorname{sgn} c_1 = \operatorname{sgn} c_3 = \operatorname{sgn}(d_1 + d_2)$, and, therefore, putting $w' = y^{d_1} x^{c_2} y^{d_2} x^{c_1+c_3}$, we get $e(w') = e(w) + (d_1 + d_2) c_1$.

2. $e(w) < 0$.

If $c_2d_1 > 0$ or $(d_1+d_2)c_3 > 0$ or $c_3d_2 > 0$, then we set $w' = x^{c_1+c_2}y^{d_1+d_2}x^{c_3}$ or $w' = x^{c_1+c_3}y^{d_1}x^{c_2}y^{d_2}$ or $w' = x^{c_1}y^{d_1}x^{c_2+c_3}y^{d_2}$, respectively, and we obtain $e(w') = e(w)-c_2d_1$ or $e(w') = e(w)-(d_1+d_2)c_3$ or $e(w') = e(w)-$ $-c_3d_2$, respectively.

If $c_2d_1 < 0$, $c_3d_2 < 0$ and $c_3d_1 < 0$, then $-\operatorname{sgn} c_2 = \operatorname{sgn} d_2$, and we put $w' = x^{c_1}y^{d_1+d_2}x^{c_2+c_3}$. Thus $e(w') = e(w)+c_2d_2$.

If $c_1d_1 < 0$, then we put $w' = y^{d_1}x^{c_1+c_2}y^{d_2}x^{c_3}$. Hence $e(w') = e(w)+$ $+c_1d_1$.

If $c_1d_1 > 0$, $(d_1+d_2)c_3 < 0$ and $c_3d_1 > 0$, then $\operatorname{sgn} c_1 = \operatorname{sgn} c_3$, and we set $w' = y^{d_1}x^{c_2}y^{d_2}x^{c_1+c_3}$. Hence $e(w') = e(w)+(d_1+d_2)c_1$.

B. Let $w = y^{d_1}x^{c_1}y^{d_2}x^{c_2}y^{d_3}$. We are going to prove that there exists a word w' of length not greater than 4 and such that $a(w') = a(w)$, $b(w') = b(w)$ and $|c(w)| < |c(w')|$. We assume that not all exponents in w are equal to 0. Consider two cases for $e(w) = c_1d_1+c_2d_1+c_2d_2$.

1. $e(w) > 0$.

If $c_1d_1 < 0$ or $c_2d_2 < 0$ or $(c_1+c_2)d_1 < 0$, then put $w' = x^{c_1}y^{d_1+d_2}x^{c_2}y^{d_3}$ or $w' = y^{d_1}x^{c_1+c_2}y^{d_2+d_3}$ or $w' = x^{c_1}y^{d_2}x^{c_2}y^{d_1+d_3}$, respectively. This yields $e(w') = e(w) - c_1d_1$ or $e(w') = e(w) - c_2d_2$ or $e(w') = e(w) - (c_1+c_2)d_1$, respectively.

If $c_1d_1 > 0$, $c_2d_1 > 0$ and $c_2d_2 > 0$, then we set $w' = y^{d_1+d_2}x^{c_1+c_2}y^{d_3}$. Hence we obtain $\operatorname{sgn} c_1 = \operatorname{sgn} d_2$, and, therefore, $e(w') = e(w)+c_1d_2$.

If $c_2d_3 > 0$, then we put $w' = y^{d_1}x^{c_1}y^{d_2+d_3}x^{c_2}$. Thus $e(w') = e(w)+$ $+c_2d_3$.

If $c_2d_3 < 0$, $c_2d_1 < 0$ and $(c_1+c_2)d_1 > 0$, then we set $w' = y^{d_1+d_3}x^{c_1}y^{d_2}x^{c_2}$. Hence $\operatorname{sgn} d_1 = \operatorname{sgn} d_3$, and, therefore, $e(w') = e(w)+(c_1+c_2)d_3$.

2. $e(w) < 0$.

If $c_1d_1 > 0$ or $c_2d_2 > 0$ or $(c_1+c_2)d_1 > 0$, then it suffices to put $w' = x^{c_1}y^{d_1+d_2}x^{c_2}y^{d_3}$ or $w' = y^{d_1}x^{c_1+c_2}y^{d_1+d_3}$ or $w' = x^{c_1}y^{d_2}x^{c_2}y^{d_1+d_3}$, respectively. This yields $e(w') = e(w)-c_1d_1$ or $e(w') = e(w)-c_2d_2$ or $e(w')$ $= e(w)-(c_1+c_2)d_1$, respectively.

If $c_1d_1 < 0$, $c_2d_2 < 0$ and $c_1d_2 < 0$, then we set $w' = y^{d_1+d_2}x^{c_1+c_2}y^{d_3}$. This yields $e(w') = e(w)+c_1d_2$.

If $c_2d_3 < 0$, then we put $w' = y^{d_1}x^{c_1}y^{d_2+d_3}x^{c_2}$. Hence $e(w') = e(w)+$ $+c_2d_3$.

If $c_2d_3 > 0$, $c_2d_1 > 0$ and $(c_1+c_2)d_1 < 0$, then we shall set $w' =$ $y^{d_1+d_3}x^{c_1}y^{d_2}x^{c_2}$. Hence $\operatorname{sgn} d_1 = \operatorname{sgn} d_3$ and $e(w') = e(w)+(c_1+c_2)d_3$.

Now by induction on the length $2t$ of the word $w = x^{c_1}y^{d_1}\ldots x^{c_t}y^{d_t}$, we see that there exists a word w' of length not greater than 4 and such that $a(w') = a(w)$, $b(w') = (bw)$ and $|e(w)| \leqslant |e(w')|$. Of course, the exponents of w' have the required property and the lemma follows.

LEMMA 3. *For k, l, n such that $n > 0$ and $|k|+|l| \leqslant n$, we have*

$$M_n(k, l) = \max\{\max_D(\varphi\psi), \max_D(kl-\varphi\psi)\} \quad \textit{and} \quad m_n(k, l) = kl - M_n(k, l),$$

where $D = \{(\varphi, \psi)\colon |\varphi|+|k-\varphi|+|\psi|+|l-\psi| \leqslant n\}$. Moreover,

$$M_n(-k, -l) = M_n(l, k) = M_n(k, l) \tag{1}$$

and

$$M_n(-k, l) = M_n(k, l) - kl. \tag{2}$$

Proof. It follows from Lemma 2 that $M_n(k, l) = e(w)$, where w is a word of length not greater than 4, that is $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2}$ or $w = y^{b_2}x^{a_2}y^{b_1}x^{a_1}$, and the equalities $a_1+a_2 = k$ and $b_1+b_2 = l$ hold. This means that w is either of the form

$$x^{k-a_2}y^{b_1}x^{a_2}y^{l-b_1} = x^k y^l [y, x]^{a_2 b_1}$$

or

$$y^{l-b_1}x^{a_2}y^{b_1}x^{k-a_2} = x^k y^l [y, x]^{kl-a_2 b_1}.$$

Since w belongs to F^n, we have $|k-a_2|+|a_2|+|b_1|+|l-b_1| \leqslant n$. Similarly,

$$\begin{aligned} m_n(k, l) &= \min\{\min_D(\varphi\psi), \min_D(kl-\varphi\psi)\} \\ &= -\max\{-\min_D(\varphi\psi), -\min_D(kl-\varphi\psi)\} \\ &= -\max\{\max_D(-\varphi\psi), -kl+\max_D(\varphi\psi)\} \\ &= -\max\{-kl+\max_D(kl-\varphi\psi), -kl+\max_D(\varphi\psi)\} \\ &= kl - \max\{\max_D(\varphi\psi), \max_D(kl-\varphi\psi)\} \\ &= kl - M_n(k, l). \end{aligned}$$

Equalities (1) are evident and we have to check only (2). Let $\varphi' = -\varphi$ and $D' = \{(\varphi', \psi)\colon |\varphi'|+|k-\varphi'|+|\psi|+|l-\psi| \leqslant n\}$. We have

$$\begin{aligned} M_n(-k, l) &= \max\{\max_D(\varphi\psi), \max_D(-kl-\varphi\psi)\} \\ &= \max\{\max_{D'}(-\varphi'\psi), -kl+\max_{D'}(\varphi'\psi)\} \\ &= \max\{\max_{D'}(\varphi'\psi)-kl, \max_{D'}(-\varphi'\psi)+kl-kl\} \\ &= \max\{\max_{D'}(\varphi'\psi), \max_{D'}(kl-\varphi'\psi)\}-kl \\ &= M_n(k, l)-kl. \end{aligned}$$

This completes the proof.

LEMMA 4. *Let k, l, n be integers satisfying the inequalities $0 \leqslant k \leqslant n/2$ and $k \leqslant l \leqslant (n+k)/3$. Then*

$$\Lambda_n(k,l) = \begin{cases} \left(\dfrac{n+k+l}{4}\right)^2 & \text{if } n+k+l \equiv 0 \pmod 4, \\ \left(\dfrac{n-1+k+l}{4}\right)^2 & \text{if } n+k+l \equiv 1 \pmod 4, \\ \left(\dfrac{n+k+l}{4}\right)^2 - \dfrac{1}{4} & \text{if } n+k+l \equiv 2 \pmod 4, \\ \left(\dfrac{n-1+k+l}{4}\right)^2 - \dfrac{1}{4} & \text{if } n+k+l \equiv 3 \pmod 4. \end{cases}$$

Let k, l, n be integers satisfying $0 \leqslant k \leqslant n/2$ and $(n+k)/3 \leqslant l \leqslant n-k$, then

$$M_n(k,l) = \begin{cases} \dfrac{n+k-l}{2} l & \text{if } n+k-l \equiv 0 \pmod 2, \\ \dfrac{n-1+k-l}{2} l & \text{if } n+k-l \equiv 1 \pmod 2. \end{cases}$$

Proof. First, we show that if $k \geqslant 0$ and $l \geqslant 0$, then

$$M_n(k,l) = \max_D (\varphi\psi),$$

where the set D is as in Lemma 3. To do this, suppose that $k, l, a_0', b_0' \geqslant 0$ and $a_0, b_0 \leqslant 0$. Observe that if $(a_0, b_0) \in D$, then $(k-a_0, \max\{b_0, l\}) \in D$, and, similarly, if $(a_0', b_0') \in D$, then $(\max\{a_0', k\}, l-b_0') \in D$. We suppose now that $\min_D(\varphi\psi) = a_0 b_0$. This gives

$$\begin{aligned} \max_D(\varphi\psi) + \min_D(\varphi\psi) &\geqslant (k-a_0)\max\{b_0, l\} + a_0 b_0 \\ &= k\max\{b_0, l\} + a_0(b_0 - \max\{b_0, l\}) \geqslant kl. \end{aligned}$$

Similarly, if $\min_D(\varphi\psi) = a_0' b_0'$, then we obtain

$$\begin{aligned} \max_D(\varphi\psi) + \min_D(\varphi\psi) &\geqslant (l-b_0')\max(a_0', k\} + a_0' b_0' \\ &= l\max\{a_0', k\} + b_0'(a_0' - \max\{a_0', k\}) \geqslant kl. \end{aligned}$$

Hence $\max_D(\varphi\psi) \geqslant kl - \min_D(\varphi\psi)$ and, consequently, $M_n(k,l) = \max_D(\varphi\psi)$.

Now it is not hard to check that the function $f(x,y) = xy$ takes on the set $\{(x,y)\colon |x|+|k-x|+|y|+|l-y| \leqslant n\}$ the maximum value

$$\max f(x,y) = \begin{cases} \left(\dfrac{n+k+l}{4}\right)^2 & \text{if } 0 \leqslant k \leqslant \dfrac{n}{2}, \max\{0, 3k-n\} \leqslant l \leqslant \dfrac{n+k}{3}, \\ \dfrac{n+k-l}{2} l & \text{if } 0 \leqslant k \leqslant \dfrac{l}{2}, \ \dfrac{n+k}{3} \leqslant l \leqslant n-k, \\ \dfrac{n+l-k}{2} k & \text{if } \dfrac{n}{3} \leqslant k \leqslant n, \ 0 \leqslant l \leqslant \min\{n-k, 3k-n\}. \end{cases}$$

LEMMA 5. *Let G be the free nilpotent group of class 2 on two generators x, y. Let $F = \{x, x^{-1}, y, y^{-1}, 1\}$. Then, for every n, we have $|F^n| \geqslant cn^4$ and $|F^n - F^{n-1}| \leqslant c'n^3$, where c, c' are constants and $c \neq 0$.*

Proof. From Lemmas 1 and 3 we infer that

$$|F^n| = \sum_{|k|+|l| \leqslant n} \big(2M_n(k,l) - kl + 1\big).$$

Let

$$\begin{aligned} \mathrm{I}_n &= \{(k,l)\colon 0 \leqslant k < n, 0 < l \leqslant n-k\}, \\ \mathrm{II}_n &= \{(k,l)\colon -n \leqslant k < 0, 0 \leqslant l \leqslant n+k\}, \\ \mathrm{III}_n &= \{(k,l)\colon -n < k \leqslant 0, -n-k \leqslant l < 0\}, \\ \mathrm{IV}_n &= \{(k,l)\colon 0 < k \leqslant n, -n+k \leqslant l \leqslant 0\}. \end{aligned}$$

Observe that

$$\sum_{|k|+|l| \leqslant n} kl = 0$$

and that, for $k, l \geqslant 0$, by Lemma 4, we have $M_n(k,l) - kl \geqslant 0$. In the following estimation we use also (1) and (2) of Lemma 3:

$$\begin{aligned} \frac{1}{2}|F^n| &\geqslant \sum_{|k|+|l| \leqslant n} M_n(k,l) \\ &= M_n(0,0) + \sum_{\mathrm{I}_n} M_n(k,l) + \sum_{\mathrm{II}_n} M_n(k,l) + \sum_{\mathrm{III}_n} M_n(k,l) + \sum_{\mathrm{IV}_n} M_n(k,l) \\ &\geqslant 4 \sum_{\mathrm{I}_n} M_n(k,l) - 2 \sum_{\mathrm{I}_n} kl \geqslant 2 \sum_{\mathrm{I}_n} kl \\ &= \frac{1}{12}(n-1)n(n+1)(n+2) \geqslant cn^4 \quad \text{for some } c \neq 0. \end{aligned}$$

Similarly, we have

$$\frac{1}{2}|F^n - F^{n-1}| = \sum_{\mathrm{I}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) + \sum_{\mathrm{II}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) +$$

$$+ \sum_{\mathrm{III}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) + \sum_{\mathrm{IV}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) +$$

$$+ M_n(0,0) - M_{n-1}(0,0) + \frac{1}{2} \sum_{|k|+|l|=n} \big(1 + 2M_n(k,l)\big).$$

Now, Lemma 4 gives $M_n(0,0) - M_{n-1}(0,0) \leqslant n^2/16$. Using (1) and (2) from Lemma 3, we have

$$\sum_{\mathrm{II}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) = \sum_{\mathrm{III}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big)$$

$$= \sum_{\mathrm{IV}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) = \sum_{\mathrm{I}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big).$$

We have also

$$\sum_{|k|+|l|=n} \big(1 + M_n(k,l)\big) \leqslant 4n + 8\sum_{k=0}^{n-1} M_n(k, n-k) + 4\sum_{k=0}^{n-1} k(n-k)$$

$$\leqslant 4n + \frac{n(n-1)(n+1)}{6} + 8\sum_{k=0}^{n-1} \left(\frac{n}{2}\right)^2 \leqslant 8n^3.$$

Hence

$$\frac{1}{8}|F^n - F^{n-1}| \leqslant \sum_{\mathrm{I}_{n-1}} \big(M_n(k,l) - M_{n-1}(k,l)\big) + 2n^3$$

$$\leqslant 2n^3 + 2\sum_{k=0}^{(n-1)/2} \big(M_n(k, n-1-k) - M_{n-1}(k, n-1-k)\big) +$$

$$+ 2\sum_{k=0}^{(n-2)/2} \sum_{l=k}^{n-k-2} \big(M_n(k,l) - M_{n-1}(k,l)\big)$$

$$\leqslant 2n^3 + 2n^3 + 2\sum_{k=0}^{(n-2)/2} \left(\left(\frac{n+k+(n+k)/3}{4}\right)^2 - \frac{n+k-3-(n+k)/3}{2}\, l\right) +$$

$$+ 2\sum_{k=0}^{(n-2)/2} \sum_{l=k}^{(n+k-2)/3} \left(\frac{1}{4} + \left(\frac{n+k+l}{4}\right)^2 - \frac{n+k+l-2}{4}\, l\right) +$$

$$+2\sum_{k=0}^{(n-4)/2}\sum_{l=(n+k+4)/3}^{n-k-2}\left(\frac{n+k-l}{2}l-\frac{n+k-l-2}{2}l\right)$$

$$\leqslant 4n^3+2\sum_{k=0}^{(n-2)/2}\left(\frac{n+k}{3}\right)^2+2\sum_{k=0}^{(n-2)/2}\sum_{l=k}^{(n+k-2)/3}\frac{n+k+l}{4}+$$

$$+2\sum_{k=0}^{(n-2)/2}\sum_{l=(n+k+3)/3}^{n-k-2} l\leqslant cn^3 \quad \text{for some } c,$$

and the lemma follows.

3. Theorem. *Let H be a nilpotent group on generators h_1, h_2. Then H satisfies* (D) *with $S=\{1, h_1, h_1^{-1}, h_2, h_2^{-1}\}$.*

Proof. Suppose, first, that H is the free nilpotent group of class 2. Then, by Lemma 5, we have

$$\frac{|S^{2n}-S^{2n-1}|}{|S^n|}\leqslant\frac{cn^3}{c'n^4}\to 0.$$

Now let H' denote the commutator subgroup of H and let $e_1=\exp H$ and $e_2=\exp H'$, where by exp G we mean the minimal natural number n such that $g^n=1$ for all $g\in G$. Of course, if H is free, then $e_1=0$ and $e_2=0$.

If $e_1>0$ and $e_2>0$, then H is finite and there is nothing to prove.

If $e_1=0$ and $e_2=0$, then an easy argument shows that H is the free nilpotent group of class 2. Indeed, let $H=G/R$, where G is the free nilpotent group of class 2 on generators x, y and R is its normal subgroup. We have to show that $R=1$. If $r=x^a y^b[y,x]^c\in R$, then $r^{-1}x^{-1}rx=[y,x]^b\in R$ and $r^{-1}y^{-1}ry=[y,x]^a\in R$. Since $e_2=0$, we get $a=b=0$, and, therefore, $c=0$. Thus $R=1$.

Suppose $e_1=0$ and $e_2>0$. Let $H=G/R$, where G is the free nilpotent group of class 2. Observe that $(G')^{e_2}\subset R$. If $|k|+|l|+4e_2\leqslant n$, then all elements of the form $x^k y^l[y,x]^c(G')^{e_2}$, $0\leqslant c<e_2$, are in S^{n-1}. Thus

$$|S^n-S^{n-1}|\leqslant e_2|\{(k,l)\colon\ n-1-4e_2\leqslant k+l\leqslant n\}|\leqslant 4e_2(4e_2+1)(n+1).$$

Let $A=\{x^i y^{n-i}R\colon\ 0\leqslant i<n\}$ and $B=\{x^{-n+i}y^iR\colon\ 0\leqslant i<n\}$. Clearly, $A, B\subset S^n-S^{n-1}$. We show that $|A|=n$ or $|B|=n$. Suppose, to the contrary, that $|A|<n$ and $|B|<n$. Then $x^a y^{-a}\in R$ and $x^b y^b\in R$ for some $a\neq 0$ and $b\neq 0$. This yields $[y,x]^a, [y,x]^b\in R$. Since $e_2(H)=e_2$, e_2 must divide a and b. From this we infer that both $x^{ab}y^{-ab}$ and $x^{ab}y^{ab}$ belong to R. Thus x^{2ab} and y^{2ab} are in R. But this is impossible, because of $e_1(G)=0$. Thus $|A|=n$ or $|B|=n$. Therefore, $|S^n-S^{n-1}|\geqslant n$.

Since

$$|S^n|=|S|+\sum_{i=2}^{n}|S^i-S^{i-1}|,$$

we have $|S^n| \geqslant \frac{1}{2}(n-1)(n-2)$. Therefore,

$$\frac{|S^{2n} - S^{2n-1}|}{|S^n|} \leqslant \frac{cn}{c'n^2}$$

for some constants c, c', and the theorem follows.

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