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FREE $\mathfrak{m}$-PRODUCTS OF LATTICES. II*

BY

GEORGE GRÄTZER AND DAVID KELLY (WINNIPEG, MANITOBA)

This paper is a continuation of Part I which also appeared in this journal([1]). The definitions, notations and conventions of Part I remain in force. In particular, $\mathfrak{m}$ is always an infinite regular cardinal, and in an $\mathfrak{m}$-lattice, every nonempty subset of cardinality less than $\mathfrak{m}$ has a join and meet. The Whitman conditions $({}_\wedge \mathrm{W})$, $({}_\vee \mathrm{W})$, $(\mathrm{W}_\wedge)$ and $(\mathrm{W}_\vee)$ refer to Theorem 1.1, the Structure Theorem for Free $\mathfrak{m}$-Products.

After some introductory remarks, we begin with Section 3, where we prove that $F_{\mathfrak{m}}(3)$ is not $\mathfrak{m}^+$-complete. In Section 4, we show that free $\mathfrak{m}$-products are only rarely $\mathfrak{m}^+$-complete.

Let $\mathfrak{m}' < \mathfrak{m}$ be an infinite regular cardinal and let $\mathscr{L} = (L_i \mid i \in I)$ be a family of $\mathfrak{m}$-lattices. Clearly, $\mathscr{L}$ is also a family of $\mathfrak{m}'$-lattices. If K is the free $\mathfrak{m}'$-product of $\mathscr{L}$ and L is the free $\mathfrak{m}$-product of $\mathscr{L}$, then the natural $\mathfrak{m}'$-homomorphism $\varphi\colon K \to L$ (that maps each L_i identically) is one-to-one. Hence, K is an $\mathfrak{m}'$-sublattice of L. However, K is not necessarily an intact sublattice of L. For example, if $\mathfrak{m}' = \aleph_0$, $\mathfrak{m} = \aleph_1$, and $\mathscr{L} = (L_0, L_1)$ with $L_0 = \omega + 1$ and $L_1 = \{e\}$, then $\bigvee(n \wedge e \mid n < \omega) = \omega \wedge e$ in K, whereas the two sides of this equality represent distinct elements in L. (The Structure Theorem of Section 1 can be used to verify these facts.) In Section 5, we show this example to be typical.

For any poset X, we investigate the natural (one-to-one) $\mathfrak{m}'$-homomorphism

$$\psi\colon CF_{\mathfrak{m}'}(X) \to CF_{\mathfrak{m}}(X)$$

that maps X identically. In Section 6, we show that both φ (of the previous paragraph) and ψ preserve coverings; e.g., if $a \prec b$ in K, then $a \prec b$ in L.

3. $F_{\mathfrak{m}}(3)$: Fixed points and completeness. Let $f = f(y)$ be a unary algebraic $\mathfrak{m}$-function of $F_{\mathfrak{m}}(3)$. Equivalently, f is an $\mathfrak{m}$-polynomial in

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$\{y, x_0, x_1, x_2\}$ where y is a variable (indeterminate) and x_0, x_1, x_2 are the free generators of $F_{\mathfrak{m}}(3)$. We interpret f as a function from $F_{\mathfrak{m}}(3)$ to itself. If $f(a) = a$ for some $a \in F_{\mathfrak{m}}(3)$, then a is called a *fixed point* of f.

The topic of fixed points is certainly of independent interest. For example, what conditions guarantee that a unary algebraic function of $F(3)$ has (does not have) a fixed point? The function f in the following theorem has been considered by Ph. M. Whitman [22], R. P. Dilworth [5], and P. Crawley and R. A. Dean [3]. However, the result is new, even for $\mathfrak{m} = \aleph_0$.

THEOREM 3.1. *If*

$$f(y) = (((((y \wedge x_0) \vee x_1) \wedge x_2) \vee x_0) \wedge x_1) \vee x_2,$$

then $f\colon F_{\mathfrak{m}}(3) \to F_{\mathfrak{m}}(3)$ *does not have a fixed point.*

Proof. (All subscripts are to be taken modulo 3.) For $i = 0, 1, 2$, we define:

$$f_i(y) = (((((y \wedge x_i) \vee x_{i+1}) \wedge x_{i+2}) \vee x_i) \wedge x_{i+1}) \vee x_{i+2}$$

and, dually,

$$g_i(y) = (((((y \vee x_i) \wedge x_{i+1}) \vee x_{i+2}) \wedge x_i) \vee x_{i+1}) \wedge x_{i+2}.$$

If any one of these six functions has a fixed point, then each one does. For example, if a is a fixed point of f_0, then $a \wedge x_0$ is a fixed point of g_1. By way of contradiction, let us assume that these six functions each have a fixed point. Let a be the fixed point of lowest rank, which by symmetry and duality, we can assume to be a fixed point of $f = f_0$.

Clearly, $a \in [(x_0 \wedge x_1) \vee x_2,\ ((x_2 \vee x_0) \wedge x_1) \vee x_2]$. Consequently, we have $x_0 \nleq a$, $x_1 \parallel a$ and $x_2 < a$. From $a = f(a)$ it now follows that a is join-reducible.

Let $p = \bigvee T$ be an $\mathfrak{m}$-polynomial of minimum rank that represents a and is in normal form (see Jónsson [15] or Grätzer and Kelly [9], and the proof of Theorem 1.12). Throughout the rest of this proof, f and g_2 (written on the right) represent the maps from $P_{\mathfrak{m}}(\{x_0, x_1, x_2\})$ to itself defined by the formulas given above. Also, for any $\mathfrak{m}$-polynomial q, let qh denote

$$(((q \wedge x_0) \vee x_1) \wedge x_2) \vee x_0.$$

By Corollary 4 of [15], since p is a normal representation and $p \equiv (ph \wedge x_1) \vee x_2 = pf$, $T = T_0 \cup T_1$, where $T_0 = \{t \in T \mid t \subseteq ph \wedge x_1\}$ and $T_1 = \{t \in T \mid t \subseteq x_2\}$. Consider the valid inequality:

$$ph \wedge x_1 \subseteq \left(\bigvee T_0\right) \vee x_2.$$

Note that the right-hand side represents a. Thus, $(_\wedge W)$ would imply that $x_0 \leqslant a$ or $x_1 \leqslant a$, a contradiction. Also, $ph \wedge x_1 \subseteq x_2$ implies that $a \leqslant x_2$, which is also untrue. Therefore, $(W_\vee)$ applies and $ph \wedge x_1 \subseteq q$ for some $q \in T_0$.

Consequently, $ph \wedge x_1 \equiv q$. Therefore, $pf \equiv q \vee x_2$. Since $p \equiv pf$, we obtain $qg_2 \equiv q$. Thus, q represents a fixed point b of g_2 and b has lower rank than a. This contradiction completes the proof of the theorem.

The next two results were proved for $\mathfrak{m} = \aleph_0$ and free lattices by Ph. M. Whitman [22]. Our proofs are based on his ideas and on Theorem 3.1.

PROPOSITION 3.2. *Let $L = CF_{\mathfrak{m}}(X)$ for a poset X, and let T be a subposet of L that is isomorphic to the initial ordinal $\mathfrak{m}$. If $b = \sup T$ exists in L, then for any $a \in L$,*

$$\sup\{a \wedge t \mid t \in T\} = a \wedge b.$$

Proof. Let $c \in L$ satisfy $a \wedge t \leqslant c$ for all $t \in T$. It suffices to show that $a \wedge b \leqslant c$. We show this by induction on the rank of c. This is trivial if $c \in X$ or c is an $\mathfrak{m}$-meet. Thus, we can assume that $c = \bigvee S$. Consider the $\mathfrak{m}$ valid inequalities:

$$a \wedge t \leqslant \bigvee S, \qquad t \in T.$$

Let $T' = \{t \in T \mid t \leqslant c\}$. If $|T'| = \mathfrak{m}$, then T' is cofinal in T, whence $b \leqslant c$. If $a \leqslant c$, we are also done. Therefore, by (W), we can assume that there is an $s \in S$ such that $|\{t \in T \mid a \wedge t \leqslant s\}| = \mathfrak{m}$. This means that $a \wedge t \leqslant s$ for all $t \in T$. By induction, $a \wedge b \leqslant s \leqslant c$, completing the proof.

Remark. Proposition 3.2 does not claim that, say, $L = F_{\aleph_1}(3)$ is continuous. In fact, failures occur in L for chains of cardinality $\aleph_0$ and $\aleph_1$. However, Proposition 3.2 does imply that $CF_{\aleph_0}(X)$ is continuous, generalizing a result of Whitman [22].

THEOREM 3.3. *$F_{\mathfrak{m}}(3)$ is not $\mathfrak{m}^+$-complete.*

We first present a simple proof under the Generalized Continuum Hypothesis (G.C.H.). As usual, the cardinal $2^{\mathfrak{m}}$ denotes $(2^{\mathfrak{n}} \mid \mathfrak{n} < \mathfrak{m})$. Note that $\mathfrak{m} \leqslant 2^{\mathfrak{m}}$. Under G.C.H., $2^{\mathfrak{m}} = \mathfrak{m}$. Since $F_{\mathfrak{m}}(\mathfrak{m})$ is isomorphic to an $\mathfrak{m}$-sublattice of $F_{\mathfrak{m}}(3)$ by Crawley and Dean [3], we have $|F_{\mathfrak{m}(3)}| = 2^{\mathfrak{m}}$. If $|X| = \mathfrak{m}$, it is well known that 2^X, the poset of all subsets of X, contains a subposet Y with $|Y| = 2^{\mathfrak{m}}$ such that any complete lattice that contains Y as a subposet also contains 2^X as a subposet. If $2^{\mathfrak{m}} = \mathfrak{m}$, $|Y| = \mathfrak{m}$, then and therefore, by [3], $F_{\mathfrak{m}}(3)$ contains Y as a subposet. Since $|2^X| = 2^{\mathfrak{m}} > \mathfrak{m} = |F_{\mathfrak{m}}(3)|$, $F_{\mathfrak{m}}(3)$ cannot contain 2^X. Therefore $F_{\mathfrak{m}}(3)$ is not complete.

Proof of Theorem 3.3. Let $L = F_{\mathfrak{m}}(3)$, let f be as in Theorem 3.1 and inductively define a_α $(\alpha < \mathfrak{m})$ by:

$a_0 = x_2$;
$a_{\beta+1} = f(a_\beta)$;
$a_\beta = \bigvee (a_\alpha \mid \alpha < \beta)$ if β is a limit ordinal.

P. Crawley and R. A. Dean [3] showed that if $\alpha < \beta < \mathfrak{m}$, then $a_\alpha < a_\beta$. (Theorem 3.1 could replace part of their argument.) Let $T = \{a_\alpha \mid \alpha < \mathfrak{m}\}$. We shall show that T does not have a supremum in L.

Suppose the contrary and let $b = \sup T$. By three applications of Proposition 3.2,

$$\sup\{f(t) \mid t \in T\} = f(b).$$

Thus, $b = f(b)$, contradicting Theorem 3.1.

4. When are free $\mathfrak{m}$-products complete? Let C_n denote an n-element chain. The lattices $C_2 * C_2$ and $C_1 * C_4$ were described by H. L. Rolf [18] (see also Grätzer [6]); both of these infinite lattices are complete. If $\mathfrak{m}$ is uncountable, we show in [10] that $C_2 *_{\mathfrak{m}} C_2$ (resp., $C_1 *_{\mathfrak{m}} C_4$) is complete and is obtained by adding 14 elements to $C_2 * C_2$ (resp., $C_1 * C_4$). In particular, $C_2 *_{\mathfrak{m}} C_2$ and $C_1 *_{\mathfrak{m}} C_4$ are countable for all $\mathfrak{m}$.

To show that an $\mathfrak{m}$-lattice L is not $\mathfrak{m}^+$-complete, it suffices by Theorem 3.3 to show that $F_{\mathfrak{m}}(3)$ is a closure sublattice of L (since a closure sublattice of an $\mathfrak{m}^+$-complete lattice is itself $\mathfrak{m}^+$-complete). Theorem 2.2(β) says that K is a closure sublattice of L whenever L is a completely free $\mathfrak{m}$-lattice generated by a poset and K is the $\mathfrak{m}$-sublattice of L $\mathfrak{m}$-generated by a subset of cardinality less than $\mathfrak{m}$. Consequently,

Proposition 4.1. *If X is a poset containing a 3-element antichain, then $F_{\mathfrak{m}}(3)$ is isomorphic to a closure sublattice of $CF_{\mathfrak{m}}(X)$, and consequently, $CF_{\mathfrak{m}}(X)$ is not $\mathfrak{m}^+$-complete.*

We shall need the following

Lemma 4.2. *Let $\mathfrak{m}' \leqslant \mathfrak{m}$ be an infinite regular cardinal. Let L be an $\mathfrak{m}$-lattice that satisfies $(\mathrm{W}_{\mathfrak{m}})$ and let $X \subseteq L$ with $0 < |X| < \mathfrak{m}'$. If X $\mathfrak{m}'$-generates an $\mathfrak{m}'$-sublattice isomorphic to $CF_{\mathfrak{m}'}(X)$, then X $\mathfrak{m}$-generates an $\mathfrak{m}$-sublattice isomorphic to $CF_{\mathfrak{m}}(X)$.*

Proof. Let K be the $\mathfrak{m}$-sublattice of L $\mathfrak{m}$-generated by X. Let $\emptyset \neq S \subseteq X$ and assume that $x \leqslant \bigvee S$ in L for some $x \in X$. Since $|S| < \mathfrak{m}'$, $x \leqslant s$ for some $s \in S$ by the Structure Theorem for $CF_{\mathfrak{m}'}(X)$. Dually, if $\bigwedge S \leqslant x$ for some $x \in X$, then $s \leqslant x$ for some $s \in S$. Since K satisfies $(\mathrm{W}_{\mathfrak{m}})$, it now follows from Jónsson [15] that K is isomorphic to $CF_{\mathfrak{m}}(X)$.

We now generalize the main result of Grätzer and Kelly [8].

Theorem 4.3. *A free $\mathfrak{m}$-product of more than two $\mathfrak{m}$-lattices is never $\mathfrak{m}^+$-complete. If A and B are $\mathfrak{m}$-lattices, then $L = A *_{\mathfrak{m}} B$ is $\mathfrak{m}$-complete iff, up to isomorphism, $A = B = C_2$ or $\{A, B\} = \{C_1, C_n\}$ for $n \in \{1, 2, 3, 4\}$.*

Proof. If L is the free $\mathfrak{m}$-product of $(L_i \mid i \in I)$ with $|I| \geqslant 3$, then choose one element in each of three L_i's and apply Proposition 2.1 to conclude that $F_{\mathfrak{m}}(3)$ is a closure sublattice of L. Consequently, L is not $\mathfrak{m}^+$-complete. Since the lattice $C_1 * C_n$ is finite for $n = 1, 2, 3$, it is certainly complete. We have already mentioned that, by [10], $C_2 *_{\mathfrak{m}} C_2$ and $C_1 *_{\mathfrak{m}} C_4$ are complete.

If A is not a chain, then $F(2) \subseteq A$, and by Proposition 2.1, $F(2) *_{\mathfrak{m}} C_1 \cong F_{\mathfrak{m}}(3)$ is a closure sublattice of L. Hence, we can assume that both A and

B are chains. If $|A| = 1$ and $|B| \geqslant 5$, then $C_1 *_{\mathfrak{m}} C_5$ is a closure sublattice of L. Otherwise, we can assume that $|A| \geqslant 2$ and $|B| \geqslant 3$ and therefore, that $C_2 *_{\mathfrak{m}} C_3$ is a closure sublattice of L. H. L. Rolf [18] showed that both $C_1 * C_5$ and $C_2 * C_3$ contain $F(3)$ as a sublattice. By Lemma 4.2, both $C_1 *_{\mathfrak{m}} C_5$ and $C_2 *_{\mathfrak{m}} C_3$ contain $F_{\mathfrak{m}}(3)$ as an $\mathfrak{m}$-sublattice. Hence, by Theorem 2.2(β), $F_{\mathfrak{m}}(3)$ is a closure sublattice of each. Consequently, L contains $F_{\mathfrak{m}}(3)$ as a closure sublattice whenever A and B are not as in the statement of the theorem. This completes the proof of the theorem.

COROLLARY 4.4. *If L is the free $\mathfrak{m}$-product of at least two $\mathfrak{m}$-lattices, then the following five conditions are equivalent.*

(i) *L is complete.*

(ii) *L is $\mathfrak{m}^+$-complete.*

(iii) *All $\mathfrak{m}^+$-joins exist in L.*

(iv) *L does not contain $F_{\mathfrak{m}}(3)$ as an $\mathfrak{m}$-sublattice.*

(v) *L does not contain $F_{\mathfrak{m}}(3)$ as a closure sublattice.*

Moreover, if $\mathfrak{m}$ is uncountable, another equivalent condition is:

(vi) *L is countable.*

Proof. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are trivial, while (iii) $\Rightarrow$(v) follows by Theorem 3.3. The implication (v)$\Rightarrow$(i) follows from the proof of Theorem 4.3. By Theorem 2.2 (β), (iv) and (v) are equivalent whenever L is a free $\mathfrak{m}$-product of finite chains (which L must be in order to be complete).

We can prove the following generalization of Proposition 4.1 using the techniques used in proving Theorem 4.3. (Disjoint union of posets is denoted by $+$.)

THEOREM 4.5. *If X is a poset that contains $C_1+C_1+C_1$, C_2+C_3 or C_1+C_5, then $F_{\mathfrak{m}}(3)$ is isomorphic to a closure sublattice of $CF(X)$, and consequently, $CF_{\mathfrak{m}}(X)$ is not $\mathfrak{m}^+$-complete,*

5. Failure of intactness upon increasing $\mathfrak{m}$. In the introduction to Part II, we gave an example of finite lattices L_0 and L_1 where $L_0 * L_1$ is not an intact sublattice of $L_0 *_{\aleph_1} L_1$. Two further examples are given by $L_0 = L_1 = C_2$ and $L_0 = C_1$, $L_1 = C_4$. We now indicate why $C_2 * C_2$ is not an intact sublattice of $C_2 *_{\mathfrak{m}} C_2$ for uncountable $\mathfrak{m}$. Let $a_1 < a_2$ and $b_1 < b_2$ be the two chains. Set $u_0 = a_1$ and $v_0 = b_1$; inductively define (for $n > 1$)

$$u_n = a_1 \vee (a_2 \wedge v_{n-1}) \quad \text{and} \quad v_n = b_1 \vee (b_2 \wedge u_{n-1}).$$

Let $S = \{u_n \wedge v_n \mid n < \omega\}$. In $C_2 * C_2$,

$$\sup S = (a_1 \vee b_1) \wedge a_2 \wedge b_2,$$

while the two sides differ in $C_2 *_{\mathfrak{m}} C_2$. Similarly, $C_1 * C_4$ is not an intact sublattice of $C_1 *_{\mathfrak{m}} C_4$ for uncountable $\mathfrak{m}$.

THEOREM 5.1. *Let* $\mathfrak{m}'$ *be an infinite regular cardinal with* $2^{\mathfrak{m}'} < \mathfrak{m}$ *and let* $\mathscr{L}$ *be a family of at least two* $\mathfrak{m}$*-lattices. Let* K *be the free* $\mathfrak{m}'$*-product of* $\mathscr{L}$ *and* L *be the free* $\mathfrak{m}$*-product of* $\mathscr{L}$. K *is an intact sublattice of* L *iff one of the following two conditions holds*:

(i) $\mathfrak{m}' = \aleph_0$ *and* K *is finite* (*that is,* $\mathscr{L}$ *is* (C_1, C_1), (C_1, C_2), *or* (C_1, C_3));

(ii) $\mathfrak{m}' > \aleph_0$ *and* K *is complete* (*so* $\mathscr{L}$ *consists of two chains as described in Theorem* 4.3).

Although, it seems reasonable that Theorem 5.1 would remain valid if the hypothesis $2^{\mathfrak{m}'} < \mathfrak{m}$ were replaced by $\mathfrak{m}' < \mathfrak{m}$, we are unable to prove this. Our proof of this theorem requires the existence of a nonzero meet-reducible element a of $F_{\mathfrak{m}}(3)$ which does not cover any element of $F_{\mathfrak{m}}(3)$. (We also say that a has no lower cover.) In fact, we can show

LEMMA 5.2. *Let* L *be a lattice satisfying* (W) *and* $(\mathrm{SD}_\wedge)$, *and assume that* $a_0, a_1, b_0, b_1 \in L$. *If* $a = a_0 \wedge a_1 \leqslant b_0 \vee b_1$, $a \| b_i$ *and* $a_i \nleqslant a \vee b_j$ $(i, j = 0, 1)$, *then* a *is meet-reducible and has no lower cover in* L.

Proof. Clearly, a is meet-reducible. Thus, a can have at most one lower cover c by (W). Assume that such c exists. Let us suppose that $a \leqslant b_i \vee c$ for $i = 0$ or 1. Since $a \nleqslant b_i$, $a \nleqslant c$ and $a_0 \wedge a_1 \leqslant b_i \vee c$, it follows by (W) that $a_j \leqslant b_i \vee c$ for $j = 0$ or 1. Consequently, $a_j \leqslant a \vee b_i$, contrary to assumption. We conclude that $a \| b_i \vee c$ $(i = 0, 1)$. Therefore, $c = a \wedge (b_0 \vee c) = a \wedge (b_1 \vee c)$. By $(\mathrm{SD}_\wedge)$, $c = a \wedge (b_0 \vee b_1 \vee c) \leqslant a$. With this contradiction, the proof is complete.

If x_0, x_1, x_2 are the generators of $F_{\mathfrak{m}}(3)$, then $(x_0 \vee x_1) \wedge \wedge (x_0 \vee x_2) \wedge (x_1 \vee x_2)$, the upper median, has no lower cover. (Take $b_0 = x_0$ and $b_1 = x_1$ in Lemma 5.2.) Since $F_{\mathfrak{m}}(3)$ contains $F_{\mathfrak{m}}(\mathfrak{m})$ by [3], this argument yields $\mathfrak{m}$ elements of $F_{\mathfrak{m}}(3)$ that do not have any lower covers. We do not know whether $F(3)$ contains an element that has no lower cover and no upper cover.

Proof of Theorem 5.1. If (i) or (ii) holds, then $K = L$. If $\mathfrak{m}' = \aleph_0$ and $\mathscr{L}$ is (C_2, C_2) or (C_1, C_4), then we have already observed that K is not an intact sublattice of L. If K is $(\mathfrak{m}')^+$-complete, then by Theorem 4.3, (i) or (ii) must be satisfied. We can therefore assume that K is not $(\mathfrak{m}')^+$-complete. Hence, by Corollary 4.4, there is a three-element subset Y of K such that

(a) K', the $\mathfrak{m}'$-sublattice of K generated by Y, is isomorphic to $F_{\mathfrak{m}'}(3)$.

(b) K' is a closure sublattice of K.

Moreover, by Lemma 4.2, the $\mathfrak{m}$-sublattice of L generated by Y is isomorphic to $F_{\mathfrak{m}}(3)$. Let a be a nonzero meet-reducible element of $F_{\mathfrak{m}'}(3)$ that has no lower cover, and let S be the set of all elements $s \in F_{\mathfrak{m}'}(3)$ satisfying $s < a$. Clearly, $0 < |S| < \mathfrak{m}$ and $a = \sup S$ in $F_{\mathfrak{m}'}(3)$. Identifying K' with $F_{\mathfrak{m}'}(3)$, we conclude from (b) that $a = \sup S$ in K. Since a is meet-reducible in L, $a \neq \bigvee S$ in L. Hence, K is not an intact sublattice of L.

COROLLARY 5.3. *If $\mathfrak{m}'$, K and L are as in Theorem* 5.1, *then the following five conditions are equivalent.*

(i) *K is an intact sublattice of L.*

(ii) *All existing $(\mathfrak{m}')^+$-joins in K are preserved in L.*

(iii) *All existing $(\mathfrak{m}')^+$-meets in K are preserved in L.*

(iv) *$K = L$.*

(v) *K and L are isomorphic.*

Proof. Since the equivalence of (i), (ii), (iii), and (iv) follows from the proof of Theorem 5.1, it remains to show that (v) implies (iv). Assume that K and L are isomorphic. In particular, K is $(\mathfrak{m}')^+$-complete. If $\mathfrak{m}'$ is uncountable, then we are done by Theorem 4.3. Therefore, we assume that $\mathfrak{m}' = \aleph_0$. If $\mathcal{L}$ is (C_2, C_2) or (C_1, C_4), then K and L are not isomorphic (see [10]). Thus, by Theorem 4.3, we can assume that $\mathcal{L}$ is (C_1, C_n) for $n = 1, 2$, or 3. Hence, $K = L$, completing the proof.

6. Preservation of small intervals upon increasing $\mathfrak{m}$. Throughout this section, $\mathfrak{m}'$ is an infinite regular cardinal and $\mathfrak{m}' < \mathfrak{m}$. We need to generalize the usual concept of breadth. An $\mathfrak{m}$-lattice has *breadth* $\mathfrak{m}'$ iff whenever $T \subseteq L$ with $0 < |T| < \mathfrak{m}$, there is $T' \subseteq T$ with $0 < |T'| < \mathfrak{m}'$ for which $\bigvee T' = \bigvee T$, and dually. Clearly, for uncountable $\mathfrak{m}$, an $\mathfrak{m}$-lattice L has breadth $\aleph_0$ iff every chain in L is finite. The following theorem, the main result of this section, says that increasing $\mathfrak{m}$ does not add any new elements to small intervals.

THEOREM 6.1. *Let $\mathfrak{m}' < \mathfrak{m}$ be an infinite regular cardinal. For $i \in I$, let L_i be an $\mathfrak{m}$-lattice of breadth $\mathfrak{m}'$. Let K be the free $\mathfrak{m}'$-product of $\mathcal{L} = (L_i \mid i \in I)$ and let L be the free $\mathfrak{m}$-product of $\mathcal{L}$. If $a < b$ in K and $|[a, b]_K| < \mathfrak{m}'$, then $[a, b]_K = [a, b]_L$.*

In the statement of Theorem 6.1, $|I| < \mathfrak{m}'$ could be assumed without any loss of generality. (If $|I| > \mathfrak{m}'$, then $|[a, b]_K| \geqslant \mathfrak{m}'$ always holds.)

The most familiar example of a small interval $[a, b]$ is a prime interval; i.e., $a \prec b$. We start with a proof of Theorem 6.1 in the special case: $\mathfrak{m}' = \aleph_0$, $|L_i| = 1$ for all $i \in I$, and $a \prec b$. By the previous remark, we can assume that $K = F(n)$ and $L = F_{\mathfrak{m}}(n)$ for some positive integer n. By McKenzie [17], there is a congruence on K whose classes are closed intervals, finite in number, such that a and b are in different classes. Clearly, this congruence induces an $\mathfrak{m}$-congruence on L that has the same properties – in fact, the classes have the same bounds. Let c (resp., d) be the upper (resp., lower) bound of the class containing a (resp., b). Since both $a = b \wedge c$ and $b = a \vee d$ hold in K, they hold in L too. This means that $a \prec b$ in L.

The following two lemmas lead to the Approximation Theorem from which Theorem 6.1 follows easily.

LEMMA 6.2. *Let $\mathcal{L} = (L_i \mid i \in I)$, K and L be as in Theorem 6.1. Let $T \subseteq L$ with $0 < |T| < \mathfrak{m}$, let $a \in K$, and let us assume that $a \leqslant \bigvee T$ in L. Under these* *assumptions, there is a subset T' of T with $0 < |T'| < \mathfrak{m}'$ such that $a \leqslant \bigvee T'$.*

Proof. If $a^{(i)} \leqslant (\bigvee T)_{(i)}$ for some $i \in I$, then $a \leqslant \bigvee (t_{(i)} \mid t \in T) = \bigvee (t_{(i)} \mid t \in T')$ for some $T' \subseteq T$ with $0 < |T'| < \mathfrak{m}'$ since L_i has breadth $\mathfrak{m}'$. If $a = \bigvee S$ with $0 < |S| < \mathfrak{m}'$ and each element of S has lower rank than a has, then we can assume by induction that, for each $s \in S$, there is $T_s \subseteq T$ with $0 < T_s < \mathfrak{m}'$ such that $s \leqslant \bigvee T_s$. Obviously, $T' = \bigcup (T_s \mid s \in S)$ satisfies the conditions of the lemma. If $(\mathrm{W}_\vee)$ holds for $a \leqslant \bigvee T$, the statement is obvious. Otherwise, we can assume that $({}_\wedge\mathrm{W})$ applies, in which case the result follows easily by induction.

LEMMA 6.3. *Let $\mathscr{L} = (L_i \mid i \in I)$, K and L be as in Theorem 6.1. Let $S \subseteq K$ with $0 < |S| < \mathfrak{m}'$ and let $T \subseteq L$ with $0 < |T| < \mathfrak{m}$. Under these assumptions, there exists $T' \subseteq T$ with $0 < |T'| < \mathfrak{m}'$ such that, setting $a = \bigvee T$ and $a' = \bigvee T'$, the following two conditions hold whenever $s \in S$:*

(i) $s \leqslant a$ *iff* $s \leqslant a'$,

(ii) $a \leqslant s$ *iff* $a' \leqslant s$.

Moreover, if $J \subseteq I$ with $0 < |J| < \mathfrak{m}'$, we can also require the following for all $i \in J$:

(iii) $a_{(i)} = (a')_{(i)}$,

(iv) $a^{(i)} = (a')^{(i)}$.

Proof. Let $S_1 = \{s \in S \mid s \leqslant a\}$. If $s \in S_1$, then by Lemma 6.2, there is $T_s \subseteq T$ such that $0 < |T_s| < \mathfrak{m}'$ and $s \leqslant \bigvee T_s$. Let $S_2 = \{s \in S \mid a \nleqslant s\}$. If $s \in S_2$, then there is $s^* \in T$ such that $s^* \nleqslant s$. For each $i \in J$, choose $R_i \subseteq T$ with $|R_i| < \mathfrak{m}'$ such that $a_{(i)} = \bigvee (t_{(i)} \mid t \in R_i)$ and $a^{(i)} = \bigvee (t^{(i)} \mid t \in R_i)$.

$$T' = \bigcup (T_s \mid s \in S_1) \cup \{s^* \mid s \in S_2\} \cup \bigcup (R_i \mid i \in J)$$

satisfies the conditions of the lemma.

Let $p \in P_\mathfrak{m}(X)$ for an arbitrary set X. The (set of) *components* of p, denoted by $\mathrm{Komp}(p)$, is the subset of $P_\mathfrak{m}(X)$ that is inductively defined as follows:

(i) $\mathrm{Komp}(x) = \{x\}$ for any $x \in X$,

(ii) If p *is* $\bigvee S$ or $\bigwedge S$, then

$$\mathrm{Komp}(p) = \{p\} \cup \bigcup \{\mathrm{Komp}(s) \mid s \in S\}.$$

Note that $|\mathrm{Komp}(p)| < \mathfrak{m}$. A subset S of $P_\mathfrak{m}(X)$ is called *component-closed* if $\mathrm{Komp}(p) \subseteq S$ whenever $p \in S$. Clearly, S is component-closed iff $\bigvee T$ or $\bigwedge T$ being in S implies $T \subseteq S$. If $S \subseteq P_\mathfrak{m}(X)$, then we set $\mathrm{Komp}(S) = \bigcup \{\mathrm{Komp}(p) \mid p \in S\}$. Note that $\mathrm{Komp}(S)$ is component-closed and $|\mathrm{Komp}(S)| < \mathfrak{m}$ whenever $|S| < \mathfrak{m}$.

THEOREM 6.4 (The Approximation Theorem). *Let $\mathfrak{m}'$, $\mathscr{L} = (L_i \mid i \in I)$, K and L be as in Theorem 6.1. Let $c \in L$, $S \subseteq K$ with $|S| < \mathfrak{m}'$ and $J \subseteq I$ with $|J| < \mathfrak{m}'$. Under these assumptions, there exists $d \in K$ such that the following four conditions hold whenever $s \in S$ and $i \in J$:*

(i) $s \leqslant c$ *iff* $s \leqslant d$,
(ii) $c \leqslant s$ *iff* $d \leqslant s$,
(iii) $c_{(i)} = d_{(i)}$,
(iv) $c^{(i)} = d^{(i)}$.

Proof. Set $X = \bigcup(L_i \mid i \in I)$. Let $p \in P_{\mathfrak{m}}(X)$ represent c and let $R \subseteq P_{\mathfrak{m}'}(X)$ represent the elements of S. We can assume that R is component-closed and that $0 < |R| < \mathfrak{m}'$. We can further assume that $i \in J$ whenever $r_{(i)}$ or $r^{(i)}$ is proper for some $r \in R$ with $i \in I$. We shall define $q \in P_{\mathfrak{m}'}(X)$ to represent the desired element d. If $p \in P_{\mathfrak{m}'}(X)$, we set $q = p$. By duality and induction, we can assume that $p = \bigvee T$, and for each $t \in T$, there exists $t^* \in P_{\mathfrak{m}'}(X)$ such that the following four conditions are satisfied whenever $r \in R$ and $i \in J$:

$$r \subseteq t \quad \text{iff} \quad r \subseteq t^*,$$

$$t \subseteq r \quad \text{iff} \quad t^* \subseteq r,$$

$$t_{(i)} = (t^*)_{(i)},$$

$$t^{(i)} = (t^*)^{(i)}.$$

By Lemma 6.3, there is $T' \subseteq T$ with $0 < |T'| < \mathfrak{m}'$ such that the following four conditions are satisfied whenever $r \in R$ and $i \in J$, where $p^* = \bigvee(t^* \mid t \in T)$ and $q = \bigvee(t^* \mid t \in T')$:

(a) $$r \subseteq p^* \quad \text{iff} \quad r \subseteq q,$$

(b) $$p^* \subseteq r \quad \text{iff} \quad q \subseteq r,$$

(c) $$(p^*)_{(i)} = (q)_{(i)},$$

(d) $$(p^*)^{(i)} = (q)^{(i)}.$$

Clearly, $q \in P_{\mathfrak{m}'}(X)$. It is obvious that $p_{(i)} = q_{(i)}$ and $p^{(i)} = q^{(i)}$ whenever $i \in J$. It remains to show that the element d of K represented by q satisfies conditions (i) and (ii) of the theorem.

Let $r \in R$ and suppose that $r \subseteq p$. If (C) applies to this inequality, then $r^{(i)} \leqslant p_{(i)}$ in L_i for some $i \in J$. Since $p_{(i)} = q_{(i)}$, it follows that $r \subseteq q$. If $r = \bigvee U$ with $U \subseteq P_{\mathfrak{m}'}(X)$ and $0 < |U| < \mathfrak{m}'$, then $u \subseteq p$ for all $u \in U$. Since $u \in R$, we conclude by induction that $u \subseteq q$. (This is the crucial step in the proof.) Consequently, $r \subseteq q$. If r is an $\mathfrak{m}'$-meet and $({}_{\wedge}\mathrm{W})$ applies, we can conclude that $r \subseteq q$ by similar reasoning. Finally, we can assume that $(\mathrm{W}_{\vee})$ applies to $r \subseteq p$. This means that $r \subseteq t$ for some $t \in T$. Therefore, $r \subseteq t^* \subseteq p_*$ and, by (a), $r \subseteq q$.

Conversely, we now assume that $r \subseteq q$ with $r \in R$. If (C), $({}_{\vee}\mathrm{W})$ or $({}_{\wedge}\mathrm{W})$ applies, then we can show that $r \subseteq p$ as above. Otherwise, $r \subseteq t^*$ for some $t \in T'$. Therefore, $r \subseteq t \subseteq p$, completing the proof of (i).

We now verify (ii). Let $p \subseteq r$ with $r \in R$. For each $t \in T$, $t \subseteq r$. Thus, $t^* \subseteq r$ for $t \in T$ and $q \subseteq r$.

Conversely, let $q \subseteq r$ with $r \in R$. By (b), $p^* \subseteq r$, which implies $t^* \subseteq r$ and $t \subseteq r$ whenever $t \in T$. Thus, $p \subseteq r$. This completes the proof of the theorem.

Proof of Theorem 6.1. Suppose that $c \in [a, b]_L$ but $c \notin [a, b]_K$. For this element c and for $S = [a, b]_K$, let $d \in K$ satisfy the conditions of Theorem 6.4. Since $a, b \in S$, it follows by (i) and (ii) that $a \leqslant d \leqslant b$. Hence, $d \in S$. Therefore, $c = d$ follows by (i) and (ii), contradicting $c \notin S$. This completes the proof of the theorem.

Observe that the above proof does not use conditions (iii) and (iv) of Theorem 6.4; however, these two conditions are essential to the proof of Theorem 6.4.

The following analog of Theorem 6.4 for completely free $\mathfrak{m}$-lattices has a similar proof.

THEOREM 6.5. *Let $\mathfrak{m}' < \mathfrak{m}$ be an infinite regular cardinal and let X be a poset. Let K be $CF_{\mathfrak{m}'}(X)$ and let L be $CF_{\mathfrak{m}}(X)$. Let $c \in L$ and let $S \subseteq K$ with $|S| < \mathfrak{m}'$. Under these assumptions, there exists $d \in K$ such that the following two conditions hold whenever $s \in S$:*

(i) *$s \leqslant c$ iff $s \leqslant d$,*

(ii) *$c \leqslant s$ iff $d \leqslant s$.*

Using Theorem 6.5, we can prove:

THEOREM 6.6. *Let $\mathfrak{m}'$, K and L be as in Theorem 6.5. If $a < b$ in K and $|[a, b]_K| < \mathfrak{m}'$, then $[a, b]_K = [a, b]_L$.*

Let K and L be as in Theorem 6.1 (or 6.6). Since usually $K \neq L$, Theorem 6.1 (or 6.6) would certainly become invalid if one merely removed the cardinality restriction on $S = [a, b]_K$. We now give examples showing that even $[a, b]_L$ is not $\mathfrak{m}$-generated by $[a, b]_K$. T will denote the $\mathfrak{m}$-sublattice of L $\mathfrak{m}$-generated by $[a, b]_K$.

Let $K = C_2 * C_2$ and $L = C_2 *_{\mathfrak{m}} C_2$ ($\mathfrak{m}$ uncountable). Let $a_1 < a_2$ and $b_1 < b_2$ be the two chains. Set $a = a_1 \wedge b_1$ and $b = (a_1 \vee b_1) \wedge a_2 \wedge b_2$. In this case $|[a, b]_L - S| = 5$. If $u = \bigvee_L(u_n \mid n < \omega)$ and $v = \bigvee_L(v_n \mid n < \omega)$, where u_n and v_n $(n < \omega)$ are defined at the beginning of Section 5, then $[a, b]_L - T = \{u \wedge v, u, v, u \vee v\}$.

Let $2^{\mathfrak{m}'} < \mathfrak{m}$, $K = F_{\mathfrak{m}'}(3)$ and $L = F_{\mathfrak{m}}(3)$. Let $a = x_0 \wedge x_1 \wedge x_2$ and $b = (x_0 \vee x_1) \wedge (x_0 \vee x_2) \wedge (x_1 \vee x_2)$, the upper median. Let $c = \bigvee_L [a, b]_K$. Clearly, $c < b$. $M = [a, c]_L \cup \{b\}$ is an $\mathfrak{m}$-sublattice that includes S. Therefore, $T \subseteq M$. However, M, and therefore T, does not include $(x_0 \vee c) \wedge (x_1 \vee c)$.

If $\mathfrak{m}' = \aleph_0$, K is as in Theorem 6.6 and $a < b$ in K, then requiring all chains in K between a and b to be finite will force $[a, b]_K$ to be finite. In fact, we shall show that this even holds in some interesting cases for Theorem 6.1. A subset D of a lattice L is *disjoint* iff $x \wedge y = u \wedge v$ whenever

$x, y, u, v \in L$ with $x \neq y$ and $u \neq v$. A lattice L satisfies the $\aleph_0$-*disjointness condition* iff any disjoint subset of L is finite. The first half of the following result is due to B. Jónsson and R. P. Dilworth (see [14]).

PROPOSITION 6.7. *If L is a lattice that satisfies* ($SD_\wedge$) *or the $\aleph_0$-disjointness condition, then the following two conditions are equivalent*:

(i) *L is finite,*

(ii) *every chain in L is finite.*

Proof. If L satisfies (ii) and L is infinite, then by Whaley [20], L contains M_ω, the countably infinite modular lattice of length two. However, M_ω fails both ($SD_\wedge$) and the $\aleph_0$-disjointness condition, completing the proof.

Consider the following two types of lattices:

(1) completely free m-lattices;

(2) free products of lattices satisfying the $\aleph_0$-disjointness condition.

Lattices of the first type satisfy ($SD_\wedge$) by Theorem 1.13. Lattices of the second type satisfy the $\aleph_0$-disjointness condition by Adams and Kelly [2].

Let L be a lattice of one of these types and let $a < b$ in L. By Proposition 6.7, if every chain between a and b in L is finite, then the interval $[a, b]$ in L is finite. It is not known whether this last statement remains true if L is allowed to be a free $\aleph_1$-product of finite lattices.

We close this section with some remarks concerning covers in free m-products. A. Day [4] showed that $F(3)$ satisfies:

(*) any pair of distinct elements can be separated by a bounded homomorphism onto a finite lattice.

Any lattice L satisfying (*) is *relatively atomic*; i.e., if $a < b$ in L, then there are $c, d \in L$ such that $a \leqslant c \prec d \leqslant b$ (see [17]). Let L be a finitely generated lattice. If L is a sublattice of $F(3)$ (equivalently, projective, see [17]), then L is a closure sublattice of $F(3)$ by Theorem 2.2(β). From A. Day's result it now follows easily that L satisfies (*), an unpublished result of A. Day.

Since $F(\aleph_0) \subseteq F(3)$, it is obvious that $F(3)$ contains infinitely many incomparable covering pairs ($a \prec b$ and $c \prec d$ are *incomparable* if $a \| c$, $a \| d$, $b \| c$ and $b \| d$). We do not know whether $F_{\mathfrak{m}}(3)$ contains m incomparable covering pairs. If M_3 is the 5-element modular, non-distributive lattice, then we do not know whether $M_3 * C_1$ is relatively atomic. Of course, Theorem 6.1 does guarantee that each cover in $M_3 * C_1$ still exists in any $M_3 *_{\mathfrak{m}} C_1$.

REFERENCES

[1] M. E. Adams and D. Kelly, *Chain conditions in free products of lattices*, Algebra Universalis 7 (1977), p. 235–243.

[2] —, — *Disjointness conditions in free products of lattices*, ibidem 7 (1977), p. 245–258.

[3] P. Crawley and R. A. Dean, *Free lattices with infinite operations*, Transactions of the American Mathematical Society 92 (1959), p. 35–47.

[4] A. Day, *Splitting lattices generate all lattices*, Algebra Universalis 7 (1977), p. 163–169.

[5] R. P. Dilworth, *Lattices with unique complements*, Transactions of the American Mathematical Society 57 (1945), p. 123–154.

[6] G. Grätzer, *General Lattice Theory*, Pure and Applied Mathematics Series, New York, Basel, Berlin 1978.

[7] G. Grätzer, A. Hajnal, and D. Kelly, *Chain conditions in free products of lattices with infinitary operations*, Pacific Journal of Mathematics 83 (1979), p. 107–115.

[8] G. Grätzer and D. Kelly, *When is the free product of lattices complete?* Proceedings of the American Mathematical Society 66 (1977), p. 6–8.

[9] –, – *A normal form theorem for lattices completely generated by a subset*, ibidem 67 (1977), p. 215–218.

[10] –, – *Completely free* m*-lattices generated by posets*.

[11] G. Grätzer and H. Lakser, *Free-lattice like sublattices of free products of lattices*, Proceedings of the American Mathematical Society 44 (1974), p. 43–45.

[12] G. Grätzer, H. Lakser and C. R. Platt, *Free products of lattices*, Fundamenta Mathematicae 69 (1970), p. 233–240.

[13] G. Grätzer and J. Sichler, *Free decompositions of a lattice*, Canadian Journal of Mathematics 27 (1975), p. 276–285.

[14] B. Jónsson, *Sublattices of a free lattice*, ibidem 13 (1961), p. 256–264.

[15] – *Arithmetic properties of freely α-generated lattices*, ibidem 14 (1962), p. 476–481.

[16] – *Relatively free products of lattices*, Algebra Universalis 1 (1972), p. 362–373.

[17] R. McKenzie, *Equational bases and nonmodular lattice varieties*, Transactions of the American Mathematical Society 174 (1972), p. 1–43.

[18] H. L. Rolf, *The free lattice generated by a set of chains*, Pacific Journal of Mathematics 8 (1958), p. 585–595.

[19] Yu. I. Sorkin, *Free unions of lattices* (in Russian), Matematičeskiĭ Sbornik, Novaja Serija, 30 (1952), p. 677–694.

[20] T. P. Whaley, *Large sublattices of a lattice*, Pacific Journal of Mathematics 28 (1969), p. 477–484.

[21] Ph. M. Whitman, *Free lattices. I*, Annals of Mathematics (2) 42 (1941), p. 325–330.

[22] – *Free lattices. II*, ibidem (2) 43 (1942), p. 104–115.

UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
R3T 2N2 CANADA

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