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EXTENSIONS OF POSITIVE OPERATORS AND EXTREME POINTS. IV

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The present paper* contains complements and comments on some results given in the preceding parts [4], [5] which will be further referred to as I, II, and III.

The paper falls into two independent sections.

In Section 1 we characterize the band generated by a vector subspace of an order complete vector lattice by means of an approximation condition considered in II, Section 2. This characterization allows us to establish a common generalization of two closely related results on lattice homomorphisms obtained independently by Luxemburg and Schep [6] and the first-named author II, III (see Theorem 3 below).

In Section 2 we deal with the existence of positive extensions of a given operator the domain of which is not necessarily majorizing. Our results are related to the classical theorem of Kantorovič, its improvement given in II, and its version for spaces with an order unit given in I.

Our notation mostly follows that of I-III. The terminology we use is standard.

Throughout Y stands for an order complete real vector lattice. For a subset N of Y we denote by B_N the band generated by N. As well known, $B_N = N^{\perp\perp}$, where $N^{\perp}$ stands for the set of all elements of Y which are disjoint to every element of N. By X we denote an ordered real vector space. In Section 1 we assume X to be directed by its ordering. For the meaning of the symbols $E(T, N)$, $E(T)$, T_i, T_e, $T_i(x+)$, and $T_e(x-)$ see I, p. 279-281. The symbols S_m and $H(X, Y)$ are defined in III, p. 263-268.

1. Bands and extreme extensions. We start with a characterization of bands.

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THEOREM 1. *Let N be a vector subspace of Y and let $y \in Y$. Then $y \in B_N$ if and only if*

$$\inf\{|y-v|: v \in N\} = 0.$$

Proof. Put $P(y) = \inf\{|y-v|: v \in N\}$ for $y \in Y$. We have to prove that $P^{-1}(0) = B_N$.

The inclusion "$\subset$" follows by the Riesz decomposition theorem, as for $y_1 \in N^{\perp}$ and $y_2 \in B_N$ we have $P(y_1+y_2) = |y_1|+P(y_2)$.

As $N \subset P^{-1}(0)$, in order to prove the other inclusion, it is enough to show that $P^{-1}(0)$ is a band in Y. Observe first that P is a sublinear map of Y into Y_+, whence $P^{-1}(0)$ is a vector subspace of Y. Moreover, $P^{-1}(0)$ is an order complete sublattice as if $A \subset P^{-1}(0)$ and $y = \sup A$, then $|y-v| \leqslant (y-a)+|a-v|$, and so $P(y) \leqslant y-a$ holds for all $a \in A$.

Put $K = \{y \in Y: y_1 \leqslant y \leqslant y_2 \text{ for some } y_1, y_2 \in P^{-1}(0)\}$. It remains to show that $K = P^{-1}(0)$. Let T be the identity operator on $P^{-1}(0)$. Then, in view of I, Theorem 3, and the definition of P, we have $T \in \operatorname{extr} E(T|N, P^{-1}(0))$. Suppose $y \in K \setminus P^{-1}(0)$. Then $P(y) > 0$, and so $T_e(P(y)) > 0$. As $P^{-1}(0)$ majorizes K, it follows that there exists $S \in \operatorname{extr} E(T|N, K)$ with $S(P(y)) = T_e(P(y))$ (see II, Theorem 1 and the proof of Lemma 1). In view of I, Theorem 3, $\inf\{S(|y-v|): v \in N\} = 0$, whence $S(P(y)) = 0$, a contradiction.

As an immediate consequence, we obtain a result which clears up the strength of an assumption made in II, Corollary 3.

COROLLARY. *Suppose $w \in Y_+$. Then w is a weak order unit of Y if and only if*

$$\inf\{|y-tw|: t \in R\} = 0$$

for each $y \in Y$.

Remark 1. The Corollary can also be obtained without referring to I and II. In this case it is enough to apply the representation theorem for M-spaces ([2], 4.3.9) in the final part of the proof of Theorem 1.

THEOREM 2. *Let M be a vector subspace of X with $M = M_+ - M_+$ and let $T \in L_+(M, Y)$. Then*

$$\operatorname{extr} E(T) = \operatorname{extr}\big(E(T) \cap L_+(X, B_{T(M)})\big).$$

Proof. Let $S \in \operatorname{extr} E(T)$ and $x \in X$. Then, by III, Theorem 2, $\inf\{S_m(x-z): z \in M\} = 0$. Hence

$$\inf\{|S(x)-S(z)|: z \in M\} = 0,$$

and so, in view of Theorem 1, $S(x) \in B_{T(M)}$. This proves the inclusion "$\subset$".

The other inclusion follows from the fact that $E(T) \cap L_+(X, B_{T(M)})$ is an extreme subset of $E(T)$.

The next theorem gives a more precise form to Theorem 3 of III. Part (b) also covers a recent result of Luxemburg and Schep ([6], Theorem 4.1) obtained independently of II and III.

THEOREM 3. *Let M be a vector subspace of X with $M = M_+ - M_+$ and let $T \in H(M, Y)$. Then*

(a) $\operatorname{extr} E(T) = E(T) \cap H(X, B_{T(M)})$.

(b) *If M is majorizing, then* $\operatorname{extr} E(T) = E(T) \cap H(X, Y)$.

Proof. According to III, Theorem 3 (a), $\operatorname{extr} E(T) \subset H(X, Y)$. This yields the inclusion "$\subset$" of (b) and, in view of Theorem 2 above, the corresponding inclusion of (a). The other inclusion of (a) is a consequence of III, Theorem 3 (b), and Theorems 1 and 2 above. The other inclusion of (b) follows from (a) as for $S \in E(T)$ we have $S(X) \subset B_{T(M)}$.

Remark 2. Clearly, the analogue of Theorem 2 for set functions holds true (cf. I, Section 2).

2. Remarks on the existence of positive extensions. The classical theorem of Kantorovič (see, e.g., I, Theorem 1) assumes that the domain of a positive operator T to be extended is majorizing. This yields, in particular, that $T_e > -\infty$. As easily seen, the latter condition is necessary for the existence of positive extensions of T. It is also sufficient in spaces with order unit (I, (ii)), and so in finite-dimensional spaces (cf. [3], (12.4)). In general, however, it does not suffice as shown by an der Heiden ([1], the Example). We shall give another example to the same effect.

Example 1. Let (Ω, Σ, μ) be a non-atomic probability space and denote by $L_0(\mu)$ the vector lattice of real-valued measurable functions on Ω. Put $M = \{t1_\Omega \colon t \in R\}$ and define $T \colon M \to R$ by $T(t1_\Omega) = t$. Then $T_e(x) = \operatorname{ess\,sup} x$ for $x \in L_0(\mu)$, and so $T_e(x) > -\infty$. However, $E(T) = \emptyset$ since, by a well-known theorem of Nikodym, there are no non-zero (linear) functionals on $L_0(\mu)$ which are continuous with respect to the topology of measure convergence, and each positive functional on $L_0(\mu)$ would be continuous ([2], 3.5.6).

In contrast with the example above, we note the following

THEOREM 4. *Let M be a vector subspace of X and let $T \in L_+(M, Y)$. Then the following three conditions are equivalent:*

(i) $E(T) \neq \emptyset$.

(ii) *There exists a sublinear map $P \colon X \to Y$ such that $P(x) \leqslant T_e(x)$ for each $x \in X$.*

(iii) *There exists a superlinear map $Q \colon X \to Y$ such that $Q(x) \leqslant T_e(x)$ for each $x \in X$.*

Proof. Clearly, (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). The converse to the first implication follows from a generalized version of the Hahn-Banach theorem ([2], 2.5.7) and Remark 2 of III.

We shall prove that (iii) $\Rightarrow$ (ii). To this end put

$$P(x) = \inf\{T_e(x_1) - Q(-x_2)\colon x_1, x_2 \in X \text{ and } x_1 + x_2 = x\}$$

for $x \in X$. Since $Q(0) = 0$ and $T_e(0) = 0$, we have $P \leqslant T_e$ and $P < \infty$. Moreover, $T_e - Q \geqslant 0$ implies $P(0) = 0$. Hence, as P is, clearly, subadditive, $P(x) \geqslant -P(-x)$, and so $P(x) > -\infty$ for all $x \in X$.

Remark 3. For $Y = R$, the implication (iii) $\Rightarrow$ (i) of Theorem 4 also follows from a result of Klee ([3], Theorem (12.2)).

Next we give an example where $E(T) \neq \emptyset$ but $\operatorname{extr} E(T) = \emptyset$. This example is in contrast with Theorem 1 of II according to which $\operatorname{extr} E(T) \neq \emptyset$ if the domain of T is majorizing.

Example 2 (cf. [2], p. 162, Remark 1). Let (Ω, Σ, μ), M, and T be as in Example 1. We regard M as a subspace of $L_p(\mu)$, where $1 \leqslant p < \infty$. Then, according to a well-known representation theorem, $E(T)$ can be identified with the set

$$\Big\{f \in L_q(\mu)_+ \colon \int_\Omega f\,d\mu = 1\Big\},$$

where q is the exponent conjugate to p. Now, as μ is non-atomic by assumption, it is easily seen that $E(T)$ has no extreme points.

We close with an example showing that Theorem 2 of I fails when R is replaced by R^2.

Example 3. Let X be an ordered vector space with an order unit, let M be a subspace of X, and let $T \in L_+(M, R)$. Put $\tilde{T} = (T, T)$. Then $\tilde{T} \in L_+(M, R^2)$, where R^2 is equipped with the usual ordering, and $\tilde{T}_i(x+) = \big(T_i(x+), T_i(x+)\big)$ and $\tilde{T}_e(x-) = \big(T_e(x-), T_e(x-)\big)$ for all $x \in X$. Assume for some $x_0 \in X$ we have $T_i(x_0+) < T_e(x_0-)$ but there is no $S \in E(T)$ with $S(x_0) = T_i(x_0+)$ (see I, the Example). Take $t \in R$ with $T_i(x_0+) < t \leqslant T_e(x_0-)$. We have $\big(T_i(x_0+), t\big) \in \,]\tilde{T}_i(x_0+), \tilde{T}_e(x_0-)[$ but there is no $V \in E(\tilde{T})$ with $V(x_0) = \big(T_i(x_0+), t\big)$.

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