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ON MAXIMAL AND COMPLETE REGIONS

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1. Introduction. In this paper* the relationship between bounded maximal regions and regions complete with respect to the Carathéodory metric is obtained. An interesting characterization of complete regions is given.

Maximal regions are those which are natural boundaries of a single-valued bounded analytic function. They are considered in [3]. Complete regions are studied in [4].

2. Definitions. Let X be a bounded region in the plane. $H^\infty(X)$ will denote the set of all single-valued bounded analytic functions on X.

Let $f \in H^\infty(X)$. If X_1 is a region such that $X_1 \cap X \neq \emptyset$ and if there is a function $f_1 \in H^\infty(X_1)$ such that $f(z) = f_1(z)$ for $z \in X_1 \cap X$, we say that f can be *extended* to X_1.

A boundary point x of X is said to be *removable* if, for every $f \in H^\infty(X)$, there exists a neighbourhood N_x of x such that f can be extended to N_x. An *essential* boundary point is one that is not removable. If every boundary point is essential, we say that X is *maximal*.

Let $x, y \in \Delta(0;1)$, the unit disc. We write

$$\left|\frac{x-y}{1-\bar{x}y}\right| = [x, y].$$

For $x, y \in X$, we define

$$d(x,y) = \sup\left\{\frac{1}{2}\log\frac{1+[f(x),f(y)]}{1-[f(x),f(y)]};\ f \in H^\infty(X) \text{ and } f\colon X \xrightarrow{\text{onto}} \Delta(0;1)\right\}.$$

This defines the Carathéodory metric on X. We say that X is *complete* if it is complete with respect to the Carathéodory metric. Necessary and sufficient conditions under which X is complete are given in [4].

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3. Results. By Theorem 15 of [3], Rudin showed that maximal regions are precisely those whose boundaries are natural boundaries of a single-valued bounded analytic function. Since the Carathéodory metric is defined in terms of bounded analytic functions, it seems reasonable to conjecture that X is complete if and only if it is maximal. We will show that this is not the case.

We now show that every complete region must be maximal. We give two proofs. The first can be generalized to higher dimensions. The second uses a result of Gamelin and Garnett [1]. It is shorter but does not immediately generalize.

THEOREM 1. *Let X be a bounded region in the plane. If X is complete, then X is maximal.*

Proof. Suppose X is not maximal. Then, by Theorem 11 of [3], there is a unique maximal region X_1 containing X. Now, X_1 is the smallest maximal region containing X. Consider the map $T\colon H^\infty(X_1) \to H^\infty(X)$ given by $Tf = f|_X$, f restricted to X. Since X_1 is the unique maximal region containing X, the map T is onto. It is, clearly, 1-1 and linear. Since $H^\infty(X_1)$ and $H^\infty(X)$ are Banach spaces, T^{-1} is continuous by the inverse mapping theorem. Hence there is a constant M with $\|f\|_{X_1} \leqslant M\|f|_X\|_X$, where $\|\cdot\|$ denotes the usual supremum norm. Therefore, every function with $\|f\|_X \leqslant 1$ extends to a function $\tilde{f}$ on X_1 with $\|\tilde{f}\|_{X_1} \leqslant M$.

We now show that the Gleason part containing X does not equal X. This proves that X is not complete with respect to d (see [4]).

Let x_0 be a removable boundary point of X. Then there is a disc $\Delta(x_0; r)$, of radius $r > 0$, with $\Delta(x_0, r) \subset X_1$. If φ is a homomorphism of $H^\infty(X)$ with $\varphi(z) = x_0$, then $\varphi(f) = \tilde{f}(x_0)$ for all $f \in H^\infty(X)$.

We must show there is a point $y \in X$ such that

$$\sup_f\{|\varphi_y(f)|;\ f\in H^\infty(X),\ \|f\|_X \leqslant 1 \text{ and } \varphi(f) = 0\} < 1 \qquad \text{and} \qquad \varphi_y(z) = y.$$

Since all $f\in H^\infty(X)$ with $\|f\|_X \leqslant 1$ extend to $\tilde{f}$ on X_1 with $\|\tilde{f}\|_{X_1} \leqslant M$,

$$|\tilde{f}(z)| \leqslant \frac{M}{r}\,|z - x_0| \quad \text{for } z\in \Delta(x_0; r) \text{ and } \tilde{f}(x_0) = 0.$$

Consider the disc $\Delta(x_0; r_1)$, where $r_1 M/r < \frac{1}{2}$. Let $y \in \Delta(x_0; r_1) \cap X$. Now

$$\sup_f\{|\varphi_y(f)|;\ f\in H^\infty(X) \text{ and } \|f\|_X \leqslant 1\} = |f(y)| = |\tilde{f}(y)| \leqslant \frac{M}{r}\,|y - x_0| \leqslant \frac{Mr_1}{r} < \frac{1}{2}.$$

Hence φ is equivalent to φ_y. Consequently, X is not complete as X is not a Gleason part.

We now give a second proof based on Theorem 3.1 of [1].

Proof. By Theorem 5 of [3], x is a removable boundary point of X if and only if $\bar{\Delta}(x; r) \cap \tilde{X}$ is a Painlevé null set for some $r > 0$, where $\tilde{X}$ denotes the complement of X, and $\bar{\Delta}$ the closure of the disc. It is known that this is true if and only if $\gamma\big(\tilde{X} \cap \bar{\Delta}(x; r)\big) = 0$, where γ represents the analytic capacity.

Let

$$E_n(x) = \{x;\ 2^{-n-1} \leqslant |z-x| \leqslant 2^{-n}\}.$$

Then $E_n(x) \cap \tilde{X} \subset \tilde{X} \cap \bar{\Delta}(x; r)$ for all large enough n. By the Main Theorem of [4] and Theorem 3.1 of [1], if X is complete, then, for every boundary point x,

$$\sum_{n=1}^{\infty} 2^n \gamma\big(E_n(x) \cap \tilde{X}\big) = \infty.$$

Therefore, $\gamma\big(\tilde{X} \cap \bar{\Delta}(x; r)\big) > 0$ and hence x is not removable. Consequently, all boundary points are essential and X is maximal.

Theorem 14 of [3] states:

Let x be an essential boundary point of X. Then there is an $f \in H^\infty(X)$ whose cluster set at x consists of the entire closed unit disc, although $|f(z)| < 1$ for every $z \in X$.

We now give an analogous characterization of complete regions. In this case the cluster set at every boundary point will consist of exactly one point.

THEOREM 2. *A bounded region X is complete if and only if, for each boundary point x, there is an $f \in H^\infty(X)$ such that $\|f\| \leqslant 1$, $|f(z)| < 1$ for $z \in X$, and f extends to be continuous on $X \cup \{x\}$ with $f(x) = 1$.*

Proof. We assume, first, that such an f exists.

Suppose $\{x_n\}$ is Cauchy with respect to d. Since d is equivalent to the ordinary metric, in order to show that $\{x_n\}$ converges with respect to d, it suffices to show that $\{x_n\}$ is contained in a compact subset of X.

Suppose this is not true. Then there is a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x$, where x is a boundary point. Since $\{x_{n_k}\}$ is Cauchy, $d(x_{n_k}, x_{n_j}) \leqslant K$ for all n_k and n_j. In particular, $d(x_{n_k}, x_{n_1}) \leqslant K$. By the assumption, there exists an f with $f(x) = 1$ and $|f(z)| < 1$ for $z \in X$. Form

$$F = \frac{f - f(x_{n_1})}{1 - \bar{f}(x_{n_1}) f}.$$

Clearly, $F \in H^\infty(X)$ and $|F(x_{n_k})| \to 1$. Therefore,

$$d(x_{n_k}, x_{n_1}) \geqslant \frac{1}{2} \log \frac{1 + |F(x_{n_k})|}{1 - |F(x_{n_k})|} \to \infty.$$

This is a contradiction.

Conversely, suppose X is complete. Then, by [4], X_λ is a peak fibre for each boundary point λ. Hence there is a continuous function $\hat{f}$ on the maximal ideal space with $\hat{f}(\varphi_\lambda) = 1$, $\varphi_\lambda \epsilon X_\lambda$ and $|\hat{f}(\varphi)| < 1$ for all $\varphi \notin X_\lambda$. Therefore, if f is the pre-image of $\hat{f}$, then $f \epsilon H^\infty(X)$ and $|f(z)| < 1$. Also, the cluster set of f at λ is $\{1\}$, since $f(\varphi_\lambda) = 1$ for $\varphi_\lambda \epsilon X_\lambda$. Therefore, putting $f(\lambda) = 1$, we get an f with the desired properties.

This characterization also allows us to show that complete regions are related to regions in which the boundary points are regular (Dirichlet problem).

Definition. Suppose X is a bounded region and x_0 is a boundary point. Let f be a continuous function on the boundary of X and let μ_f be the solution of Dirichlet's problem *via* the Perron-Wiener-Brelot method with boundary values f. If for all such f we have

$$\lim_{\substack{z \to x_0 \\ z \epsilon X}} \mu_f(z) = f(x_0),$$

then x_0 is called a *regular* boundary point.

We shall use the following theorem proved in [2]:

Let X be a bounded region and let x_0 be a boundary point. If there is a positive superharmonic function ω in $\{x \epsilon X;\ |x - x_0| < \varepsilon\}$ for some $\varepsilon > 0$ and $\omega(z) \to 0$ as $z \to x_0$, then x_0 is a regular point.

THEOREM 3. *If X is complete, then every boundary point is regular.*

Proof. Let x_0 be a boundary point. By Theorem 2, there is a non-constant $f \epsilon H^\infty(X)$ with $\|f\| \leqslant 1$ and $f(x) \to 1$ as $x \to x_0$.

Since f is analytic, $-|f|$ is superharmonic. Therefore, $1 - |f|$ is superharmonic in X and $1 - |f(x)| \to 0$ as $x \to x_0$. Hence x_0 is a regular boundary point.

Summarizing, we have shown that if X is a complete region, then it must be maximal and every boundary point must be regular.

In the next section we will give examples to show that the converses of the theorems above are false. To that end we give the following definition:

Definition. Let V be an open set in $\boldsymbol{C}$. We say that $x \epsilon \partial V$ (boundary of V) is *linearly accessible* from V if there exists a straight-line segment in $V \cup \{x\}$ having x as one of its end points.

LEMMA. *Let X be a bounded region satisfying* $\operatorname{Int}\bar{X} = X$. *Then the set of linearly accessible points from* $(\bar{X})^\sim$ *is dense in* ∂X.

Proof. Since $\operatorname{Int}\bar{X} = X$, we have $\partial X = \partial \bar{X}$. Therefore, $\partial X = \partial Y$, where $Y = (\bar{X})^\sim$. Hence we need only prove that the linearly accessible points from Y are dense in ∂Y.

Suppose $x \epsilon \partial Y$ and V_x is an open set containing x. Let $\Delta(x; r) \subset V_x$ and let $y \epsilon \Delta(x; r) \cap Y$. Then there is a straight-line segment l_{yx} joining y

to x in $\Delta(x; r)$. By identifying l_{yx} with $[0, 1]$, we obtain a $y_1 \epsilon l_{yx}$ with $y_1 \epsilon \partial Y$, and $y_1 \leqslant t$ if $t \epsilon l_{yx} \cap \partial Y$. Hence $l_{yy_1} \subset Y \cup \{y_1\}$ and y_1 is linearly accessible from Y. Since $y_1 \epsilon V_x$, x is a limit point of linearly accessible points.

THEOREM 4. *Every bounded region satisfying* $\operatorname{Int} \overline{X} = X$ *is maximal.*

Proof. By [3], every linearly accessible point is essential. Since the set of essential points is closed, it follows from the Lemma that every boundary point is essential.

4. Examples. In this section we give examples of (1) a region which is maximal but not complete, and (2) a region all of whose boundary points are regular but is not complete.

Examples of the kind to be used have been used elsewhere. For both examples we use the following notation:

$$Y = \Delta(0; 1),$$

$$A = \bigcup_{n=1}^{\infty} \bar{\Delta}(x_n; r_n), \qquad \text{where } \bar{\Delta}(x_n; r_n) = \{x;\ |x - x_n| \leqslant r_n\},$$

$$1 > x_1 > x_2 > \ldots > x_n \to 0,$$

$$x_1 + r_1 < 1, \qquad x_{n+1} + r_{n+1} < x_n - r_n.$$

Take $X = Y - (A \cup \{0\})$ and observe that $\operatorname{Int} \overline{X} = X$. Hence, by Theorem 4, X is maximal.

Clearly, every boundary point of X is a peak fibre except, perhaps, at 0. However, Zalcman showed in [6] that $\{0\}$ is a peak fibre if and only if

$$\sum_{n=1}^{\infty} \frac{r_n}{x_n} = \infty.$$

Therefore, to construct a region which is not complete, we simply choose $\{x_n\}$ and $\{r_n\}$ such that

$$\sum_{n=1}^{\infty} \frac{r_n}{x_n} < \infty.$$

For example, we choose

$$x_n = \frac{3}{4 \cdot 2^n} \quad \text{and} \quad r_n = \frac{1}{100 \cdot 10^n}.$$

Clearly, $x_1+r_1 < 1$, $1 > x_1 > x_2 > \ldots \to 0$ and

$$\frac{3}{4\cdot 2^{n+1}}+\frac{1}{100\cdot 10^{n+1}} < \frac{3}{4\cdot 2^{n}}-\frac{1}{100\cdot 10^{n}},$$

$$\sum_{n=1}^{\infty}\frac{r_n}{x_n} = \frac{1}{75}\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n = \frac{1}{300}.$$

Hence X is not complete.

We can also use the same example to give a region whose boundary consists only of regular points. To show this, we use Wiener's test: $x\in\partial X$ is regular if and only if

$$\sum_{n=1}^{\infty}\frac{n}{-\log c_n} = \infty,$$

where $c_n = C[\tilde{X}\cap E_n(x)]$ and C denotes the logarithmic capacity.

Using the same $\{x_n\}$ and $\{r_n\}$ as before, we have

$$\gamma\big(\tilde{X}\cap E_n(x)\big) \geqslant \gamma(D_n),$$

where D_n denotes the line segment

$$\left[\frac{3}{2^{n+1}}-\frac{1}{10^{n+1}},\ \frac{3}{2^{n+1}}+\frac{1}{10^{n+1}}\right].$$

Therefore,

$$\gamma[B] \geqslant \frac{1}{2}\,\frac{1}{10^{n+1}} \quad \text{and} \quad C(B) \geqslant \gamma(B),$$

where $B = \tilde{X}\cap E_n(x)$. Hence

$$C(B) \geqslant \frac{1}{2}\,\frac{1}{10^{n+1}}, \quad \text{i.e.} \quad c_n \geqslant \frac{1}{2}\,\frac{1}{10^{n+1}}.$$

Clearly, $c_n \leqslant 1/2^n$. Therefore,

$$-\log 2-(n+1)\log 10 \leqslant \log c_n \leqslant -n\log 2,$$

that is

$$\log 2+(n+1)\log 10 \geqslant -\log c_n \geqslant n\log 2.$$

Therefore,

$$\frac{n}{\log 2+(n+1)\log 10} \leqslant \frac{n}{-\log c_n} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{n}{-\log c_n} = \infty.$$

Hence 0 is a regular point. This is the second example.

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