COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 2

A GAME OF FAIR DIVISION WITH CONTINUUM OF PLAYERS

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0. Let X be an object to be divided among n players. The set of players T is numbered from 1 to n. A partition $P = (A_1, A_2, \ldots, A_n)$ of X is a *fair division* if player i receiving the part A_i is "satisfied" for $i = 1, 2, \ldots, n$ (the precise mathematical meaning is given below).

A game of fair division with a finite number of players was introduced by Kuhn [4], nevertheless the problem of fair division has been considered in the literature since 1946 (cf. [2], [3], [6]–[8]).

A generalization of Kuhn's definition for an arbitrary set of players is given by the following

DEFINITION 1. The *game of fair division* is determined by

$$\langle X, \mathscr{B}_X, T, \mathscr{B}_T, \lambda, m, \{\mu_t\}_{t\in T}\rangle,$$

where X is an object to be divided, $\mathscr{B}_X$ is a σ-algebra of subsets of X, and other symbols are defined as follows:

T is a set of players and $\mathscr{B}_T$ is a σ-algebra of subsets of T; the subsets of T in $\mathscr{B}_T$ are called *coalitions*.

$\{\mu_t\}_{t\in T}$ is a family of probability measures defined on $(X, \mathscr{B}_X)$ such that for all $A \in \mathscr{B}_X$ the function $\mu_{\cdot}(A)\colon T \to [0, 1]$ is $\mathscr{B}_T$-measurable; μ_t is the individual evaluation of X by the player t.

λ is a probability measure defined on $(T, \mathscr{B}_T)$; it represents the relative size of a coalition $C \in \mathscr{B}_T$.

m is a probability measure defined on $(T, \mathscr{B}_T)$; it represents the relative power of a coalition $C \in \mathscr{B}_T$. The measure m represents also the least value acceptable for each coalition. We assume that m is absolutely continuous with respect to λ.

For the game of fair division we define a set function

$$V_C(\cdot)\colon \mathscr{B}_X \to [0, 1]$$

by

$$V_C(A) = \begin{cases} \frac{1}{\lambda(C)} \int_C \mu_t(A)\lambda(dt) & \text{if } \lambda(C) > 0, \\ 0 & \text{if } \lambda(C) = 0. \end{cases}$$

The value $V_C(A)$ is an estimation of $A \in \mathscr{B}_X$ by the coalition $C \in \mathscr{B}_T$. We say that $A \in \mathscr{B}_X$ is a *fair share* of C if $V_C(A) \geqslant m(C)$.

DEFINITION 2. A multifunction P: $T \to \mathscr{B}_X$ is called a *division* of X if
(a) $t_1 \neq t_2$ implies $P(t_1) \cap P(t_2) = \emptyset$,
(b) $\bigcup \{P(t): t \in T\} = X$,
(c) $E \in \mathscr{B}_T$ implies $P(E) \in \mathscr{B}_X$, where

$$P(E) = \bigcup_{t \in E} P(t).$$

Let $\mathscr{P}$ be the set of all divisions.

DEFINITION 3. A division $P \in \mathscr{P}$ is called *fair* (*ε-fair*) for the game $\langle X, \mathscr{B}_X, T, \mathscr{B}_T, \lambda, m, \{\mu_t\}_{t \in T}\rangle$ if

$$V_C(P(C)) \geqslant m(C) \quad (V_C(P(C)) \geqslant m(C) - \varepsilon)$$

for every coalition $C \in \mathscr{B}_T$.

In Section 1 we shall prove the existence of fair division in the finite case and in Section 2 the theorem on the existence of ε-fair division if T is the interval $[0, 1]$.

1. The finite case: $T = \{1, 2, \ldots, n\}$. We shall show that our criterion for a division to be fair coincides with the following result of Dubins and Spanier [2]:

THEOREM 1 ([2]). *Let $\mu_1, \mu_2, \ldots, \mu_n$ be non-atomic probability measures defined on $(X, \mathscr{B}_X)$ and let $m = (m_1, m_2, \ldots, m_n)$ be a vector in R^n such that $m_i \geqslant 0$, $i \in T$, and $\sum_{i \in T} m_i = 1$. Then there is a measurable partition $P = (A_1, A_2, \ldots, A_n)$ of X such that $\mu_i(A_j) = m_j$ for all $i, j \in T$.*

Indeed, without loss of generality, we may assume that $m(\{i\}) = m_i > 0$. Setting $\lambda(\{i\}) = 1/n$ for each $i = 1, 2, \ldots, n$, we obtain $m \ll \lambda$. The division P defined by $P(i) = A_i$ is measurable and, moreover, for each $C \subset \{1, 2, \ldots, n\}$ with $\lambda(C) > 0$ we have

$$V_C(P(C)) = \frac{1}{\lambda(C)} \sum_{i \in C} \mu_i(P(C))\lambda(\{i\})$$
$$= \frac{1}{\lambda(C)} \sum_{i \in C} \mu_i\big(\bigcup_{j \in C} P(j)\big)\lambda(\{i\})$$

$$= \frac{1}{\lambda(C)} \sum_{i \in C} \sum_{j \in C} \mu_i(A_j) \lambda(\{i\})$$

$$= \frac{1}{\lambda(C)} \sum_{i \in C} \lambda(\{i\}) \sum_{j \in C} m_j = \frac{1}{\lambda(C)} m(C) \sum_{i \in C} \lambda(\{i\})$$

$$= \frac{\lambda(C)}{\lambda(C)} m(C) = m(C).$$

Hence P is a fair division.

2. The infinite case: $T = [0, 1]$. Let ε be an arbitrary positive number. We shall show the existence of ε-fair division for the game with continuum of players.

LEMMA. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be non-atomic non-negative measures defined on $(X, \mathscr{B}_X)$ such that*

(1) $$|\mu_i(X) - a| \leqslant \xi, \quad a > 0,\ \xi \geqslant 0,\ \mu_i(X) > 0,\ i = 1, 2, \ldots, n.$$

Assume that $\alpha \in (0, a)$. Then there is an $A \in \mathscr{B}_X$ such that

(2) $$|\mu_i(A) - \alpha| \leqslant \frac{\alpha}{a}\xi,$$

(3) $$|\mu_i(X - A) - (a - \alpha)| \leqslant \frac{a-\alpha}{a}\xi \quad \textit{for all } i = 1, 2, \ldots, n.$$

Proof. Let us consider measures

$$\nu_i = \frac{1}{\mu_i(X)} \mu_i.$$

It follows from the Lyapunov convexity theorem (see [1] and [5]) that there exists a set $A \in \mathscr{B}_X$ such that, for all $i = 1, 2, \ldots, n$, $\nu_i(A) = \alpha/a$. Then

$$\nu_i(A) = \frac{\mu_i(A)}{\mu_i(X)} = \frac{\alpha}{a}.$$

Hence

$$\mu_i(A) = \mu_i(X) \frac{\alpha}{a}.$$

Multiplying (1) by α/a we get (2), and by $(a - \alpha)/a$ we get (3).

THEOREM 2. *Let T be the interval $[0, 1]$ and let $\mathscr{B}_T$ be the σ-algebra of Borel subsets of $[0, 1]$. Assume that $(X, \mathscr{B}_X)$ is a measurable space. Let $\{\mu_t\}_{t \in T}$ be a family of probability measures defined on $(X, \mathscr{B}_X)$ such that*

(a) *there is a dense set D in T such that μ_t is non-atomic for each $t \in D$,*
(b) *there is a number $M > 0$ such that*

$$\sup_{A \in \mathscr{B}_X} |\mu_t(A) - \mu_s(A)| \leqslant M|t-s|.$$

Let λ be the Lebesgue measure on $[0, 1]$ *and let m be a probability measure on* $[0, 1]$ *absolutely continuous with respect to λ. Then for an arbitrary $\varepsilon > 0$ there exists a division P_ε such that*

$$\left|V_C(P_\varepsilon(C)) - m(C)\right| \leqslant \varepsilon \quad \textit{for all } C \in \mathscr{B}_T.$$

(Note that (a) and (b) imply that all μ_t are non-atomic.)

Proof. Let $\varepsilon > 0$ be an arbitrary positive number. Let $K(\xi)$ be a finite subset of D such that

$$\forall t \in T\ \exists t' \in K(\xi):\ |t-t'| < \xi/M.$$

We put

$$E_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad E_{2^n}^n = \left[\frac{2^n-1}{2^n}, 1\right],$$
$$n = 1, 2, \ldots, \ k = 1, 2, \ldots, 2^n - 1.$$

It follows from the Lyapunov convexity theorem that there exists an $A_1^1 \in \mathscr{B}_X$ such that

$$\mu_t(A_1^1) = m(E_1^1) \quad \text{for all } t \in K(\varepsilon/4).$$

Let $t \in T$. Then there exists $t_0 \in K(\varepsilon/4)$ such that $|t-t_0| \leqslant e/4M$. By the assumption (b), for all $t \in T$ we have

$$|\mu_t(A_1^1) - m(E_1^1)| = |\mu_t(A_1^1) - \mu_{t_0}(A_1^1)| \leqslant \varepsilon/4$$

and

$$|\mu_t(A_2^1) - m(E_2^1)| \leqslant \varepsilon/4,$$

where $A_2^1 = X - A_1^1$ and $E_2^1 = T - E_1^1$. If $m(E_2^1) = 0$ or $m(E_1^1) = 0$, we put $A_2^1 = \emptyset$ or $A_1^1 = \emptyset$, respectively. Assume that $m(E_1^1) > 0$ and $m(E_2^1) > 0$. Consider the set $K(\varepsilon/2^4)$ and the divisions

$$E_1^1 = E_1^2 \dot{\cup} E_2^2 \quad \text{and} \quad E_2^1 = E_3^2 \dot{\cup} E_4^2.$$

By the Lemma we see that for $t \in K(\varepsilon/2^4)$ the sets A_i^2 $(i = 1, 2, 3, 4)$ exist with the inequalities

$$|\mu_t(A_i^2) - m(E_i^2)| \leqslant \frac{m(E_i^2)}{m(E_1^1)} \frac{\varepsilon}{4} \quad \text{for } i = 1, 2,$$

$$|\mu_t(A_j^2) - m(E_j^2)| \leqslant \frac{m(E_j^2)}{m(E_2^1)} \frac{\varepsilon}{4} \quad \text{for } j = 3, 4,\ t \in K\left(\frac{\varepsilon}{2^4}\right).$$

Hence for all $t \in T$ we have

$$|\mu_t(A_i^2) - m(E_i^2)| \leqslant \frac{m(E_i^2)}{m(E_1^1)}\frac{\varepsilon}{4} + \frac{\varepsilon}{16} \quad \text{for } i = 1, 2$$

and

$$|\mu_t(A_j^2) - m(E_j^2)| \leqslant \frac{m(E_j^2)}{m(E_2^1)}\frac{\varepsilon}{4} + \frac{\varepsilon}{16} \quad \text{for } j = 3, 4.$$

If $m(E_j^2) = 0$ for one of $j = 1, 2, 3, 4$, we put $A_j^2 = \emptyset$. Let us assume that $m(E_j^2) \neq 0$, $E_j^2 = E_k^3 \dot\cup E_{k+1}^3$. Analogously, for $t \in K(\varepsilon/2^6)$ there exist A_k^3 and A_{k+1}^3 such that

$$|\mu_t(A_k^3) - m(E_k^3)| \leqslant \left(\frac{m(E_j^2)}{m(E_1^1)}\frac{\varepsilon}{4} + \frac{\varepsilon}{16}\right)\frac{m(E_k^3)}{m(E_j^2)}$$

and

$$|\mu_t(A_{k+1}^3) - m(E_{k+1}^3)| \leqslant \frac{m(E_{k+1}^3)}{m(E_1^1)}\frac{\varepsilon}{4} + \frac{m(E_{k+1}^3)}{m(E_j^2)}\frac{\varepsilon}{16}.$$

Hence for all $t \in T$ we get

$$|\mu_t(A_k^3) - m(E_k^3)| \leqslant \frac{m(E_k^3)}{m(E_1^1)}\frac{\varepsilon}{4} + \frac{m(E_k^3)}{m(E_j^2)}\frac{\varepsilon}{16} + \frac{\varepsilon}{64}.$$

We continue this procedure putting $A_k^n = \emptyset$ if $m(E_k^n) = 0$. To every set E_k^n we assign the set $A_k^n \in \mathscr{B}_X$ such that for all $t \in T$

$$|\mu_t(A_k^n) - m(E_k^n)| \leqslant \frac{m(E_k^n)}{m(E_{i_1}^1)}\frac{\varepsilon}{4} + \frac{m(E_k^n)}{m(E_{i_2}^2)}\frac{\varepsilon}{16} + \ldots + \frac{m(E_k^n)}{m(E_{i_{n-1}}^{n-1})}\frac{\varepsilon}{2^{2n-2}} + \frac{\varepsilon}{2^{2n}},$$

where $E_k^n \subset E_{i_{n-1}}^{n-1} \subset \ldots \subset E_{i_1}^1$. Put

$$\varepsilon_k(n) = \frac{m(E_k^n)}{m(E_{i_1}^1)}\frac{\varepsilon}{4} + \frac{m(E_k^n)}{m(E_{i_2}^2)}\frac{\varepsilon}{16} + \ldots + \frac{\varepsilon}{2^{2n}}.$$

We shall show that

$$(4) \qquad \sum_{k=1}^{2^n} \varepsilon_k(n) < \varepsilon \quad \text{for all } n \in N.$$

We have

$$\sum_{k=1}^{2^n} \varepsilon_k(n) = \frac{\varepsilon}{4}\sum_{\{j:\, E_j^n \subset E_1^1\}} \frac{m(E_j^n)}{m(E_1^1)} + \frac{\varepsilon}{4}\sum_{\{j:\, E_j^n \subset E_2^1\}} \frac{m(E_j^n)}{m(E_2^1)} +$$
$$+ \frac{\varepsilon}{16}\sum_{\{j:\, E_j^n \subset E_1^2\}} \frac{m(E_j^n)}{m(E_1^2)} + \ldots + \frac{\varepsilon}{16}\sum_{\{j:\, E_j^n \subset E_4^2\}} \frac{m(E_j^n)}{m(E_4^2)} + \ldots +$$

$$+\frac{\varepsilon}{2^{2n-2}}\sum_{\{j:E_j^n\subset E_1^{n-1}\}}\frac{m(E_j^n)}{m(E_1^{n-1})}+\ldots$$

$$+\frac{\varepsilon}{2^{2n-2}}\sum_{\{j:E_j^n\subset E_{2^{n-1}}^{n-1}\}}\frac{m(E_j^n)}{m(E_{2^{n-1}}^{n-1})}+2^n\frac{\varepsilon}{2^{2n}}$$

$$=\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\left(\frac{\varepsilon}{16}+\ldots+\frac{\varepsilon}{16}\right)+\ldots+2^{n-1}\frac{\varepsilon}{2^{2n-2}}+2^n\frac{\varepsilon}{2^{2n}}<\varepsilon.$$

For every $n\in N$ we define a multifunction P_n: $[0, 1]\to\mathscr{B}_X$ by

$$P_n(t) = A_k^n \quad \text{whenever } t\in E_k^n.$$

We put

$$P_\varepsilon(t) = \bigcap_{n\in N} P_n(t).$$

Then for all $t\in T$ we have $P_\varepsilon(t)\in\mathscr{B}_X$. We shall prove that P_ε is a fair division. First notice that P_ε is a division. Moreover, for E_k^n and for all $m\geqslant n$ we have

$$P_\varepsilon(E_k^n) = P_m(E_k^n) = A_k^n.$$

For, let $x\in P_\varepsilon(E_k^n)$. There exists $t_0\in E_k^n$ such that, for all $m\in N$, $x\in P_m(t_0)$. Hence $x\in A_k^n$. Conversely, let $x\in A_k^n$. Then x belongs to one of the sets A_{2k-1}^{n+1}, A_{2k}^{n+1}. Assume that $x\in A_{2k-1}^{n+1}$. Continuing this procedure we get k_m for all $m\geqslant n$ such that $x\in A_{k_m}^m$. Then

$$x\in\bigcap_{m\geqslant n} A_{k_m}^m,\qquad t\in\bigcap_{m\geqslant n} E_{k_m}^n\neq\emptyset.$$

Since

$$\bigcap_{m\geqslant n} A_{k_m}^m = P_\varepsilon(t)\qquad\text{for some } t\in T,$$

we have $x\in P_\varepsilon(t)$ and $t\in E_{k_m}^m$. Therefore $x\in P_\varepsilon(t)\subset P_\varepsilon(E_k^n)$.

We shall show that for all $C\in\mathscr{B}_T$

$$P_\varepsilon(C) = \bigcup_{t\in C} P_\varepsilon(t)\in\mathscr{B}_X.$$

Put $\mathscr{K} = \{C\subset[0, 1];\ P_\varepsilon(C)\in\mathscr{B}_X\}$. Since P_ε is a partition, $\mathscr{K}$ is closed under the formation of unions and complements. Since $\mathscr{K}$ is a σ-algebra and contains the sets E_k^n generating $\mathscr{B}_T$, we have $\mathscr{B}_T\subset\mathscr{K}$.

Let $\mathscr{A}$ be the class of sets which are finite unions of the sets E_k^n:

$$\forall A\in\mathscr{A}\ \exists n\ \exists k_1, k_2, \ldots, k_m:\ A = \bigcup_{i=1}^m E_{k_i}^n,\ m\leqslant 2^n.$$

We shall prove that $|V_{B_2}(P_\varepsilon(B_1))-m(B_1)| \leqslant \varepsilon$ for any B_1 and B_2 from $\mathscr{A}$. There exists $n_0 \in N$ such that

$$B_1 = \bigcup_{j=1}^{m_1} E_{k_j}^{n_0} \quad \text{and} \quad B_2 = \bigcup_{l=1}^{m_2} E_{r_l}^{n_0}.$$

It is easy to show that for E_k^n we have

$$|V_{B_2}(P_\varepsilon(E_k^n))-m(E_k^n)| \leqslant \varepsilon_k(n), \tag{5}$$

where

$$V_{B_2}(P_\varepsilon(E_k^n)) = \frac{1}{\lambda(B_2)} \int_{B_2} \mu_t(P_\varepsilon(E_k^n))\lambda(dt) = \frac{1}{\lambda(B_2)} \int_{B_2} \mu_t(A_k^n)\lambda(dt)$$

and

$$m(E_k^n)-\varepsilon_k(n) \leqslant \mu_t(A_k^n) \leqslant m(E_k^n)+\varepsilon_k(n).$$

For, integrating all sides of the above inequalities on the set B_2 with respect to λ and dividing by $\lambda(B_2)$ we get (5).

We have

$$|V_{B_2}(P_\varepsilon(B_1))-m(B_1)| = \left|\frac{1}{\lambda(B_2)} \int_{B_2} \mu_t(P_\varepsilon(B_1))\lambda(dt)-m(B_1)\right|$$

$$= \left|\frac{2^{n_0}}{m_2} \sum_{l=1}^{m_2} \int_{E_{r_l}^{n_0}} \mu_t(P_\varepsilon(B_1))\lambda(dt)-m(B_1)\right|$$

$$= \left|\frac{2^{n_0}}{m_2} \sum_{l=1}^{m_2} \sum_{j=1}^{m_1} \int_{E_{r_l}^{n_0}} \mu_t(P_\varepsilon(E_{k_j}^{n_0}))\lambda(dt)-m(B_1)\right|$$

$$= \left|\frac{1}{m_2} \sum_{l=1}^{m_2} \sum_{j=1}^{m_1} \frac{1}{1/2^{n_0}} \int_{E_{r_l}^{n_0}} \mu_t(P_\varepsilon(E_{k_j}^{n_0}))\lambda(dt)-\frac{1}{m_2} \sum_{l=1}^{m_2} \sum_{j=1}^{m_1} m(E_{k_j}^{n_0})\right|$$

$$= \left|\frac{1}{m_2} \sum_{l=1}^{m_2} \sum_{j=1}^{m_1} \left(\frac{1}{1/2^{n_0}} \int_{E_{r_l}^{n_0}} \mu_t(A_{k_j}^{n_0})\lambda(dt)-m(E_{k_j}^{n_0})\right)\right|$$

$$\leqslant \frac{1}{m_2} \sum_{l=1}^{m_2} \sum_{j=1}^{m_1} |V_{E_{r_l}^{n_0}}(A_{k_j}^{n_0})-m(E_{k_j}^{n_0})|$$

$$\leqslant \frac{1}{m_2} \sum_{l=1}^{m_2} \sum_{j=1}^{m_1} \varepsilon_{k_j}(n_0) \leqslant \varepsilon,$$

where the last inequality follows from (4), because $m_1 \leqslant 2^{n_0}$. It is easy to verify that $V_{B_2}((P_\varepsilon(\cdot))$ is a measure. Then

$$\sup_{C \in \mathscr{B}_T} |V_{B_2}(P_\varepsilon(C))-m(C)| = \sup_{B_1 \in \mathscr{A}} |V_{B_2}(P_\varepsilon(B_1))-m(B_1)| \leqslant \varepsilon.$$

Hence for every $C \in \mathscr{B}_T$ we get

$$|V_{B_2}(P_\varepsilon(C)) - m(C)| \leqslant \varepsilon. \tag{6}$$

Now we show that for every $F \in \mathscr{B}_T$ such that $\lambda(F) > 0$ we have

$$V_F(P_\varepsilon(C)) \geqslant m(C) - \varepsilon. \tag{7}$$

Let η be any positive number. Then there exists a set $E \in \mathscr{A}$ such that

$$F \subset E \quad \text{and} \quad \frac{\lambda(E \dot{-} F)}{\lambda(E)} < \eta.$$

Now

$$\begin{aligned} V_F(P_\varepsilon(C)) &= \frac{1}{\lambda(F)} \int_F \mu_t(P_\varepsilon(C)) \lambda(dt) \\ &\geqslant \frac{1}{\lambda(E)} \Big(\int_E \mu_t(P_\varepsilon(C)) \lambda(dt) - \int_{E-F} \mu_t(P_\varepsilon(C)) \lambda(dt) \Big) \\ &\geqslant m(C) - \varepsilon - \frac{\lambda(E-F)}{\lambda(E)} \geqslant m(C) - \varepsilon - \eta, \end{aligned}$$

and therefore $V_F(P_\varepsilon(C)) \geqslant m(C) - \varepsilon - \eta$. Since the number η is arbitrary, we have (7).

Let $B \in \mathscr{A}$ and $F \in \mathscr{B}_T$ be such that $\lambda(F) > 0$, $F \subset B$, $C \in \mathscr{B}_T$, and $\lambda(C) > 0$. Then

$$\begin{aligned} V_F(P_\varepsilon(C)) &\leqslant \frac{1}{\lambda(F)} \int_B \mu_t(P_\varepsilon(C)) \lambda(dt) \\ &= \frac{\lambda(B)}{\lambda(F)} \left(\frac{1}{\lambda(B)} \int_B \mu_t(P_\varepsilon(C)) \lambda(dt) \right) \leqslant \frac{\lambda(B)}{\lambda(F)} (m(C) + \varepsilon), \end{aligned}$$

where the last inequality follows from (6).

For any positive number ξ and given C and F we can choose a $B \in \mathscr{A}$ such that

$$\left(\frac{\lambda(B)}{\lambda(F)} - 1 \right) (m(C) + \varepsilon) < \xi.$$

Then $V_F(P_\varepsilon(C)) \leqslant m(C) + \varepsilon + \xi$. Hence for all $C, F \in \mathscr{B}_T$ such that $\lambda(F) > 0$ and $\lambda(C) > 0$ we have

$$|V_F(P_\varepsilon(C)) - m(C)| \leqslant \varepsilon.$$

The inequality holds in particular for $F = C$, which completes the proof.

Remark. Assume that the set of continuous functions $\{\mu_t(A) \colon A \in \mathscr{B}_X\}$ is closed in the space $C(T)$ of continuous functions with the sup-norm. Then it is easy to see that for every $\alpha \in [0, 1]$ there exists a set $A \in \mathscr{B}_X$ such that $\mu_t(A) = \alpha$ for all $t \in T$. Hence, modifying the above proof (putting $\varepsilon = 0$) it is easy to show that there exists a fair division P such that $V_C(P(C)) = m(C)$.

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Reçu par la Rédaction le 25. 1. 1983;
en version modifiée le 14. 4. 1983