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STABILITY AND EXACTNESS

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0. Introduction. The purpose of this paper is to show an application of theorems on asymptotical stability of a stochastic semigroup in ergodic theory.

In Section 1, a necessary and sufficient condition for asymptotical stability of a stochastic semigroup of Markov operators on the Banach space of all finite, countably additive set functions is proved. This condition – the existence of an upper measure – is equivalent to the existence of a lower measure [10].

In Section 2, the construction of a stochastic semigroup corresponding to a semidynamical system is shown and the exactness is arrived at from the asymptotical stability of this semigroup.

The notion of exactness was introduced by Rokhlin [11], who proved that it implies ergodicity and mixing of all orders. The discovery that exactness of a semidynamical system may be characterized by asymptotic behavior of the corresponding stochastic semigroup is due to Lin [7]. From his result it follows that the semidynamical system $\{S^n\}$ (of the iterates of transformation S) with the probability measure m invariant under S is exact if and only if the stochastic semigroup $\{P_S^n\}$ is asymptotically stable on $L^1(m)$ (in the sense of [5]) with the stationary density equal to 1. Hence this condition may be used to verify whether the system with the given measure is exact. Another weaker condition can be found in [6]. It says that the asymptotical stability of $\{P_S^n\}$ on $L^1(m)$ (where S is a nonsingular transformation) implies the existence of a unique absolutely continuous invariant probability measure with respect to which the system $\{S^n\}$ is exact. Our proposition is that the asymptotical stability of $\{P_S^n\}$ (in our sense) implies the existence of a unique invariant probability measure for which the system $\{S^n\}$ is exact (such a system will be called *uniquely exact*).

In Section 3, examples of a uniquely exact system and an exact system which is not uniquely exact are given.

1. Asymptotical stability of a stochastic semigroup of Markov operators. Let $(X, \mathcal{A})$ be a measurable space with σ-field $\mathcal{A}$. Denote by N

$= (N(X, \mathcal{A}), \| \ \|)$ the Banach space of all finite σ-additive functions on $\mathcal{A}$ equipped with the norm

$$\|\nu\| = |\nu|(X);$$

here

$$|\nu| = \nu^+ + \nu^-,$$

$$\nu^+(A) = \nu(A \cap H), \quad \nu^-(A) = -\nu(A \cap H') \quad \text{for } A \in \mathcal{A},\ \nu \in N,$$

where H, H' is the Hahn decomposition of X for ν. The sets

$$N_+ = \{\nu \in N: \nu \geqslant 0\}$$

of all finite measures on $(X, \mathcal{A})$ and

$$N_p = \{\nu \in N_+: \|\nu\| = 1\}$$

of all probability measures on $(X, \mathcal{A})$ play a special role in this paper.

LEMMA 1.1. *If $\varkappa \in N_+$ and $\mu \in N_p$, then*

$$\|\mu - \varkappa\| = \|\varkappa\| - 1 + 2\|(\mu - \varkappa)^+\|.$$

Proof. If H, H' is the Hahn decomposition of X for $\mu - \varkappa$, then

$$(\mu - \varkappa)^-(X) = (\varkappa - \mu)(H') = \|\varkappa\| - 1 + \|(\mu - \varkappa)^+\|,$$

which, by the equality $\|\mu - \varkappa\| = (\mu - \varkappa)^-(X) + \|(\mu - \varkappa)^+\|$, completes the proof.

DEFINITION 1.1. A linear mapping P: $N \to N$ is called a *Markov operator* on N iff

$$P(N_p) \subset N_p. \tag{1.1}$$

It is easy to see that a linear mapping P is a Markov operator on N iff

$$P\nu \in N_+ \quad \text{and} \quad \|P\nu\| = \|\nu\| \quad \text{for all } \nu \in N_+. \tag{1.1a}$$

Every Markov operator on N has the following properties:

$$P\nu_1 \leqslant P\nu_2 \quad \text{for } \nu_1 \leqslant \nu_2,\ \nu_1, \nu_2 \in N, \tag{1.2}$$

$$(P\nu)^+ \leqslant P\nu^+, \quad (P\nu)^- \leqslant P\nu^- \quad \text{for } \nu \in N, \tag{1.3}$$

$$P\nu = \nu \Rightarrow P\nu^+ = \nu^+ \text{ and } P\nu^- = \nu^- \text{ for } \nu \in N, \tag{1.4}$$

$$|P\nu| \leqslant P|\nu|, \quad \|P\nu\| \leqslant \|\nu\| \quad \text{for } \nu \in N, \tag{1.5}$$

(1.6) P is uniformly continuous.

DEFINITION 1.2. A family $\{P^t: t \in T\}$ of Markov operators on N, where T is a semigroup of real positive numbers (i.e., $\emptyset \neq T \subset (0, \infty)$, $t_1 + t_2 \in T$ for

$t_1, t_2 \in T$) is called a *stochastic semigroup* on N iff

(1.7) $$P^{t_1+t_2} = P^{t_1} \circ P^{t_2} \quad \text{for } t_1, t_2 \in T.$$

DEFINITION 1.3. A stochastic semigroup $\{P^t: t \in T\}$ is called *asymptotically stable* iff there exists a unique probability measure μ^0 such that

(1.8) $$\lim_{t \to \infty} P^t \mu = \mu^0 \quad \text{for every } \mu \in N_p.$$

DEFINITION 1.4. A probability measure μ^0 is called a *stationary probability measure* for a stochastic semigroup $\{P^t: t \in T\}$ iff

(1.9) $$P^t \mu^0 = \mu^0 \quad \text{for all } t \in T.$$

PROPOSITION 1.1. *Every asymptotically stable stochastic semigroup has a unique stationary probability measure.*

Proof. Condition (1.8) implies, according to (1.6), that

$$P^{t'}(P^t \mu) \to P^{t'}(\mu^0) \quad \text{as } t \to \infty$$

and, according to (1.1), that

$$P^t(P^{t'} \mu) \to \mu^0 \quad \text{as } t \to \infty$$

for all $t' \in T$ and $\mu \in N_p$. Hence $P^{t'}(\mu^0) = \mu^0$ for all $t' \in T$. Assume that μ^1 is also a stationary probability measure. Then

$$\|\mu^1 - \mu^0\| = \|P^t \mu^1 - \mu^0\| \to 0 \quad \text{as } t \to \infty,$$

which implies that $\mu^1 = \mu^0$.

Remark 1.1. Property (1.4) implies that the stationary probability measure for an asymptotically stable stochastic semigroup is a unique (up to the sign) fixed point in the set $\{\nu \in N: \|\nu\| = 1\}$ of all operators from this semigroup (see [10]).

DEFINITION 1.5. A measure $\varkappa \in N_+$ is called an *E-upper measure* ($E \subset N$) for a semigroup $\{P^t: t \in T\}$ iff $\|\varkappa\| < 2$ and

(1.10) $$\lim_{t \to \infty} \|(P^t \nu - \varkappa)^+\| = 0 \quad \text{for all } \nu \in E.$$

Denote by $K^E = K^E(P^t: t \in T)$ the set of all E-upper measures for the semigroup $\{P^t: t \in T\}$.

Remark 1.2. If $\varkappa \in K^{N_p}(P^t: t \in T)$, then $\|\varkappa\| \geqslant 1$. If

$$\varkappa \in K^{N_p}(P^t: t \in T) \cap N_p,$$

then $\varkappa$ is a unique stationary probability measure for the semigroup $\{P^t: t \in T\}$.

The main result of this paper will be proved under the assumption that

T is a semigroup such that

$$(1.11) \qquad t_1 - t_2 \in T \quad \text{for } t_1 > t_2,\ t_1, t_2 \in T.$$

LEMMA 1.2. *For every stochastic semigroup* $\{P^t\colon t \in T\}$ *and every* $\nu \in N$ *the function* $t \to \|P^t \nu\|$ *is decreasing.*

THEOREM 1.1. *A stochastic semigroup* $\{P^t\colon t \in T\}$ *is asymptotically stable if and only if the set* $K^{N_p}(P^t\colon t \in T)$ *is nonempty (i.e., if there exists an* N_p*-upper measure for this semigroup).*

Proof. The "only if" part is obvious because the stationary probability measure of the semigroup is an N_p-upper measure for it.

The proof of the "if" part will be done in three steps.

First, we are going to show that

$$(1.12) \qquad \lim_{t \to \infty} \|P^t(\mu_1 - \mu_2)\| = 0 \quad \text{for } \mu_1, \mu_2 \in N_p.$$

Fix two arbitrary probability measures μ_1 and μ_2. For $\nu = \mu_1 - \mu_2$ we have

$$\|\nu^+\| = \|\nu^-\| = \tfrac{1}{2}\|\nu\| =: c,$$

because

$$\nu^+(X) - \nu^-(X) = \nu(X) = \mu_1(X) - \mu_2(X) = 0 \quad \text{and} \quad \|\nu\| = \nu^+(X) + \nu^-(X).$$

Assume for a moment that $c > 0$ and $\varkappa \in K^{N_p}$. Then

$$\|P^t \nu\| = c\left\|\big(P^t(\nu^+/c) - \varkappa\big) - \big(P^t(\nu^-/c) - \varkappa\big)\right\|$$
$$\leqslant c\big(\|P^t(\nu^+/c) - \varkappa\| + \|P^t(\nu^-/c) - \varkappa\|\big).$$

Since the measures ν^+/c and ν^-/c belong to N_p, there exists, according to (1.10), a $t_1 \in T$ such that for $t \geqslant t_1$

$$\left\|\big(P^t(\nu^+/c) - \varkappa\big)^+\right\| \leqslant (2 - \|\varkappa\|)/4, \qquad \left\|\big(P^t(\nu^-/c) - \varkappa\big)^+\right\| \leqslant (2 - \|\varkappa\|)/4.$$

Therefore, by Lemma 1.1 and (1.1),

$$\|P^t \nu\| \leqslant \|\nu\|\,\|\varkappa\|/2 \quad \text{for } t \geqslant t_1.$$

For $c = \|\nu\|/2 = 0$ this inequality is obvious because P^t is linear. Finally, for any $\mu_1, \mu_2 \in N_p$ we have

$$\|P^{t_1}(\mu_1 - \mu_2)\| \leqslant \|\mu_1 - \mu_2\|\,\|\varkappa\|/2.$$

In the same way we can find a time $t_2 \in T$ such that

$$\|P^{t_1 + t_2}(\mu_1 - \mu_2)\| \leqslant \|P^{t_1}\mu_1 - P^{t_2}\mu_2\|\,\|\varkappa\|/2 \leqslant \|\mu_1 - \mu_2\|\big(\|\varkappa\|/2\big)^2,$$

because P^t preserves the norm on N_+. After n steps we obtain

$$\|P^{t_1 + \ldots + t_n}(\mu_1 - \mu_2)\| \leqslant \|\mu_1 - \mu_2\|\big(\|\varkappa\|/2\big)^n,$$

where $t_1, \dots, t_n$ are suitably chosen elements from T. Now, putting $\tau_n = t_1 + \dots + t_n$, we have, by $\|\varkappa\| < 2$,

$$\lim_{n \to \infty} \|P^{\tau_n}(\mu_1 - \mu_2)\| = 0.$$

This implies, according to Lemma 1.2, that (1.12) holds.

In the second step we shall construct the least N_p-upper measure for our stochastic semigroup. First observe that $\varkappa_1 \wedge \varkappa_2 \in K^{N_p}$ for all $\varkappa_1, \varkappa_2 \in K^{N_p}$. In fact, if H, H' denotes the Hahn decomposition for $\varkappa_1 - \varkappa_2$, then

$$\begin{aligned} \|(P^t\mu - \varkappa_1 \wedge \varkappa_2)^+\| &= (P^t\mu - \varkappa_1 \wedge \varkappa_2)^+(H) + (P^t\mu - \varkappa_1 \wedge \varkappa_2)^+(H') \\ &= (P^t\mu - \varkappa_2)^+(H) + (P^t\mu - \varkappa_1)^+(H') \\ &\leqslant \|(P^t\mu - \varkappa_1)^+\| + \|(P^t\mu - \varkappa_2)^+\|. \end{aligned}$$

Write $K = \inf\{\|\varkappa\|: \varkappa \in K^{N_p}\}$. We can choose a sequence $\{\bar{\varkappa}_n\}$ of N_p-upper measures such that $\bar{\varkappa}_n \to K$. Replacing, if necessary, $\{\bar{\varkappa}_n\}$ by a sequence $\{\varkappa_n\}$ defined by

$$\varkappa_1 = \bar{\varkappa}, \quad \varkappa_{n+1} = \varkappa_n \wedge \bar{\varkappa}_{n+1}, \ n \in N,$$

we get a decreasing sequence of N_p-upper measures such that

$$\varkappa_n(X) \to K \in [1, 2).$$

Since $\|\varkappa_m - \varkappa_n\| = \varkappa_n(X) - \varkappa_m(X) < \varepsilon$ for $m \geqslant n \geqslant n_0(\varepsilon)$, there exists $\varkappa^0 \in N$ such that $\|\varkappa_n - \varkappa^0\| \to 0$. Thus $\varkappa_n \to \varkappa^0$ uniformly on $\mathscr{A}$, which implies that $\varkappa^0 \in N_+$ and $\|\varkappa^0\| = K < 2$. To show that $\varkappa^0$ is an N_p-upper measure it is enough to check that

$$\|(P^t\mu - \varkappa^0)^+\| \leqslant \|(P^t\mu - \varkappa_n)^+\| + \|\varkappa_n - \varkappa^0\|$$

for all $t \in T$ and $n \in N$. Let $\mu \in N_p$, $n \in N$ and let H_0, H_0' and H_n, H_n' denote the Hahn decompositions of X for $\mu - \varkappa^0$ and $\mu - \varkappa_n$, respectively. Then

$$\begin{aligned} \|(\mu - \varkappa^0)^+\| &= (\mu - \varkappa^0)(H_0) = (\mu - \varkappa_n)(H_0) + (\varkappa_n - \varkappa^0)(H_0) \\ &\leqslant (\mu - \varkappa_n)(H_0 \cap H_n) + \|\varkappa_n - \varkappa^0\| \\ &\leqslant (\mu - \varkappa_n)(H_n) + \|\varkappa_n - \varkappa^0\| = \|(\mu - \varkappa_n)^+\| + \|\varkappa_n - \varkappa^0\|. \end{aligned}$$

Finally, $\varkappa^0$ is the least element in K^{N_p} because it is a minimal element in K^{N_p}, and K^{N_p} is closed with respect to minimum.

In the last step we prove the asymptotical stability of our stochastic semigroup. First, observe that for every $t' \in T$ and $\varkappa \in K^{N_p}$ also $P^{t'}\varkappa \in K^{N_p}$. In fact, according to (1.11), (1.7), (1.3) and (1.1a), for $t > t'$ and $\mu \in N_p$ we have

$$\begin{aligned} \|(P^t\mu - P^{t'}\varkappa)^+\| &= \|(P^{t'}(P^{t-t'}\mu - \varkappa))^+\| \\ &\leqslant \|P^{t'}(P^{t-t'}\mu - \varkappa)^+\| = \|(P^{t-t'}\mu - \varkappa)^+\|. \end{aligned}$$

This means that $P^t \varkappa^0 \geqslant \varkappa^0$ for $t \in T$. Moreover, $P^t \varkappa^0 = \varkappa^0$, for if not, then there exists a set $A \in \mathscr{A}$ such that $P^t \varkappa^0(A) > \varkappa^0(A)$ and we have $\|P^t \varkappa^0\| > \|\varkappa^0\|$, which contradicts (1.1a). Thus $\mu^0 = \varkappa^0/\|\varkappa^0\|$ is a probability measure satisfying (1.9). Now, according to (1.12), we have

$$\|P^t \mu - \mu^0\| = \|P^t(\mu - \mu^0)\| \to 0 \text{ as } t \to \infty \quad \text{for } \mu \in N_p.$$

This completes the proof.

Remark 1.3. Assumption (1.11) in Theorem 1.1 may be replaced by

(1.13) the function $t \to \|P^t(\mu_1 - \mu_2)\|$ is decreasing for all $\mu_1, \mu_2 \in N_p$.

In fact, by Theorem 1.1 for an arbitrary $s \in T$ the semigroup $\{P^{ns}: n \in N\}$ is asymptotically stable because for the semigroup $\{ns: n \in N\}$ the condition (1.11) holds. If μ_0 is the stationary probability measure for P^s, then for a fixed $t \in T$

$$\|P^t \mu_0 - \mu_0\| = \|P^t(P^{ns} \mu_0) - \mu_0\| = \|P^{ns}(P^t \mu_0) - \mu_0\| \to 0 \quad \text{as } n \to \infty,$$

which implies $P^t \mu_0 = \mu_0$. Now, by (1.13), for arbitrary $\mu \in N_p$ we have

$$\lim_{t \to \infty} \|P^t \mu - \mu_0\| = \lim_{n \to \infty} \|P^{ns}(\mu - \mu_0)\| = 0.$$

Remark 1.4. Assumption (1.11) is essential in Lemma 1.2 and Theorem 1.1, and assumption (1.13) is essential in Remark 1.3 as the following example shows:

Example 1.1. Let $X = \{a, b\}$ and $\mathscr{A} = 2^X$. For $\nu \in N(X, \mathscr{A})$ we have

$$\nu = u\delta_a + v\delta_b, \quad \text{where } u, v \in \boldsymbol{R} \text{ and } \|\nu\| = |u| + |v|.$$

The operator π on N given by

$$\pi(u\delta_a + v\delta_b) = \big(u + (v/2)\big)\delta_a + (v/2)\delta_b$$

is a Markov operator such that for $p, q \geqslant 0$, $p + q = 1$ and $n \in N$

$$\pi^n(p\delta_a + q\delta_b) = \big(p + (1 - 1/2^n)q\big)\delta_a + (q/2^n)\delta_b \leqslant \delta_a + (1/2)\delta_b.$$

Consequently, $\delta_a + (1/2)\delta_b$ is an N_p-upper measure for the stochastic semigroup $\{\pi^n: n \in N\}$ which is asymptotically stable with the stationary probability measure δ_a. Let $\varepsilon \in (0, 1) \setminus \boldsymbol{Q}$. The set

$$\{k + l\varepsilon: k \in N, \ l \in \{1, \ldots, k\}\}$$

is a semigroup of real positive numbers for which (1.11) does not hold. Writing $P^{k+l\varepsilon} = \pi^l$ for $k \in N$, $l = 1, \ldots, k$, we get the stochastic semigroup $\{P^t: t \in T\}$ such that $\delta_a + (1/2)\delta_b$ is an N_p-upper measure for it, (1.13) does not hold (because $\|P^{k+2\varepsilon}(\delta_b - \delta_a)\| = 1/2$ and $\|P^{k+\varepsilon}(\delta_b - \delta_a)\| = 1$ for all $k \geqslant 2$) and it is not asymptotically stable (because $\|P^{n+\varepsilon}(\delta_b - \delta_a)\| = 1$ for all $n \in N$).

Now, denote by $\langle\ \rangle$ the convex closed hull, i.e., for $E \subset N$,

$$\langle E\rangle = \{\sum_{k=1}^{m} \alpha_k \mu_k \colon \sum_{k=1}^{m} \alpha_k = 1,\ \alpha_k > 0,\ \mu_k \in E,\ k = 1, \ldots, m;\ m \in N\}$$

PROPOSITION 1.2. *If* $\langle E\rangle = N_p$, *then the conditions*

(a) $$\varkappa \in K^E(P^t \colon t \in T),$$

(b) $$\varkappa \in K^{N_p}(P^t \colon t \in T)$$

are equivalent.

Remark 1.5. The assumption of the existence of an N_p-upper measure in Theorem 1.1 may be replaced by the assumption of the existence of an N_p-lower measure [10]. In that paper one may also find a proof of the analog of Proposition 1.2 for lower measures and a proof of the fact that the asymptotical stability of the subsemigroup $\{P^{ns} \colon n \in N\}$ for some $s \in T$ implies the asymptotical stability of $\{P^t \colon t \in T\}$, which will be used below.

2. Exactness. Let $(X, \mathscr{A})$ be a measurable space. A transformation $S\colon X \to X$ is called *double measurable* iff $S(A) \in \mathscr{A}$ and $S^{-1}(A) \in \mathscr{A}$ for all $A \in \mathscr{A}$. A family $\{S_t \colon t \in T\}$ of double measurable transformations is called a *semidynamical system* iff

(2.1) $$S_{t_1+t_2} = S_{t_1} \circ S_{t_2} \quad \text{for } t_1, t_2 \in T,$$

where T is a semigroup of real positive numbers. A measure $\mu^0 \in N$ is called *invariant* under $\{S_t \colon t \in T\}$ iff

(2.2) $$\mu^0(S_t^{-1}(A)) = \mu^0(A) \quad \text{for all } A \in \mathscr{A} \text{ and } t \in T.$$

Every semidynamical system $\{S_t \colon t \in T\}$ determines by the formula

(2.3) $$P_S^t v(A) = v(S_t^{-1}(A)) \quad \text{for } A \in \mathscr{A} \text{ and } t \in T$$

a stochastic semigroup $\{P_S^t \colon t \in T\}$ of Markov operators on N, which has the following properties:

(i) if $\mu \in N_+$ is concentrated on $A \in \mathscr{A}$, then $P_S^t \mu$ is concentrated on $S_t(A)$;

(ii) $\mu \in N_+$ is invariant under $\{S_t \colon t \in T\}$ if and only if μ is a fixed point of all operators from $\{P_S^t \colon t \in T\}$.

A semidynamical system $\{S_t \colon t \in T\}$ is μ^0-*exact* ([11], [5]) iff μ^0 is a probability measure on $(X, \mathscr{A})$, invariant under $\{S_t \colon t \in T\}$ and such that

(2.4) $$\lim_{t \to \infty} \mu^0(S_t(A)) = 1 \quad \text{for all } A \in \mathscr{A} \text{ such that } \mu^0(A) > 0.$$

DEFINITION 2.1. A semidynamical system $\{S_t \colon t \in T\}$ is called *uniquely exact* iff there exists a unique probability measure μ^0 such that the system $\{S_t \colon t \in T\}$ is μ^0-exact.

Unique exactness implies exactness.

PROPOSITION 2.1. *If $\{S_t\colon t\in T\}$ is a semidynamical system such that the corresponding stochastic semigroup $\{P_S^t\colon t\in T\}$ is asymptotically stable, then $\{S_t\colon t\in T\}$ is uniquely exact.*

Proof. Let μ^0 be the unique stationary probability measure for the semigroup $\{P_S^t\colon t\in T\}$. By (ii), μ^0 is the unique probability measure invariant under $\{S_t\colon t\in T\}$. For the proof of (2.4) fix $A\in\mathscr{A}$ such that $\mu^0(A)>0$. Since

$$\mu_A^0=\mu^0(\cdot\,|A)\in N_p,$$

by (i) we have

$$0\leqslant 1-\mu^0(S_t(A))\leqslant |P_S^t\mu_A^0-\mu^0|(S_t(A))\leqslant \|P_S^t\mu_A^0-\mu^0\|$$

and, consequently, $\mu^0(S_t(A))\to 1$ as $t\to\infty$.

From Theorem 1.1, Proposition 1.2, Remark 1.5, Proposition 2.1 and the property of upper variation we obtain immediately (under the assumption (1.11) or (1.13))

COROLLARY 2.1. *If there exists $s\in T$, $E\subset N_p$ and $\varkappa\in N_+$ such that $\langle E\rangle = N_p$, $\|\varkappa\|<2$ and*

$$\lim_{n\to\infty}\sup_{A\in\mathscr{A}}\big(P_S^{ns}\mu(A)-\varkappa(A)\big)=0\quad \textit{for all } \mu\in E, \tag{2.5}$$

then the semidynamical system $\{S_t\colon t\in T\}$ is uniquely exact.

3. Examples. (I) Let $X=(0,1]$ and $\mathscr{A}$ be the collection of all Borel subsets of X. Consider the transformation $S\colon X\to X$ defined by

$$S(x)=\begin{cases}2x, & x\in(0,1/2],\\ 1, & x\in(1/2,1].\end{cases}$$

The semidynamical system $\{S^n\colon n\in N\}$ of all iterates of S is uniquely exact.

Indeed, observe that for $\mu\in N_p((0,1],\mathscr{A})$ we have

$$\sup_{A\in\mathscr{A}}\big(P_S^n\mu(A)-\delta_1(A)\big)=\sup_{1\notin A\in\mathscr{A}}\mu(S^{-n}(A))\leqslant \mu\big(S^{-n}((0,1))\big)=\mu\big((0,1/2^n)\big),$$

which implies that (2.5) holds for $E=N_p$, $s=1$ and $\varkappa=\delta_1$. Since

$$\delta_1\in K^{N_p}(P_S^n\colon n\in N)\cap N_p,$$

it is, by virtue of Remark 1.2, a stationary probability measure for this semigroup, which is asymptotically stable and, according to (ii), is invariant under our semidynamical system.

(II) Let $X=[0,1]$ and $\mathscr{A}$ be the collection of Borel subsets of X. Consider the transformation $S\colon X\to X$ defined by

$$S(x)=\begin{cases}0, & x\in[0,1/4],\\ 2x-1/2, & x\in(1/4,3/4),\\ 1, & x\in[3/4,1].\end{cases}$$

The semidynamical system $\{S^n: n \in N\}$ of all iterates of S is not uniquely exact but it is δ_0-exact and δ_1-exact.

Hence the corresponding stochastic semigroup $\{P_S^n: n \in N\}$ is not asymptotically stable. However, it generates stochastic semigroups $\{{}^iP_S^n: n \in N\}$ on $L^1(\delta_i)$, $i = 0, 1$, where

$$ {}^iP_S(f) = \frac{dP_S v_f^i}{d\delta_i} \quad \text{and} \quad v_f^i(A) = \int_A f d\delta_i \quad \text{for } A \in \mathcal{A},\ f \in L^1(\delta_i), $$

which are asymptotically stable in the sense of Lasota [5]. This follows from results of Lin [7].

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