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QUASIREGULAR RINGS

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1. Introduction. In [4] we studied strongly Baer ideals in an a.c. ring. In this paper we introduce a topology on the set $\gamma(R)$ of all prime strongly Baer ideals of an a.c. ring R. It is proved that with this topology $\gamma(R)$ is a T_0-space with the family of compact open subsets of $\gamma(R)$ as a base for the open sets. It is shown that R is quasiregular if and only if the exterior of every compact open subset of $\gamma(R)$ is compact. Further, some equivalent conditions are established for $\mathcal{M}(R)$ (the space of minimal prime ideals of R with the hull-kernel topology) to be a compact and discrete space.

2. Preliminaries. For any ideal I of a commutative ring S, we put

$$I^* = \{x \in S \mid xy = 0 \text{ for all } y \in I\}.$$

I is called *nondense* if $I^* \neq \{0\}$.

According to Henriksen and Jerison [3], a commutative ring S without nonzero nilpotents is said to satisfy the *annihilator condition* or is said to be an *a.c. ring* if for any $x, y \in S$ there is $z \in S$ such that $(x)^* \cap (y)^* = (z)^*$. A commutative ring S is called *regular* if for each $s \in S$ there exists an element $x \in S$ such that $s = s^2 x$. A commutative ring S with at least one nonzero divisor is said to be a *quasiregular ring* if its classical ring of quotients is regular. It is well known that a commutative semiprime ring with at least one nonzero divisor is quasiregular if and only if for each $x \in S$ there is a $y \in S$ such that $(x)^{**} = (y)^*$ (see [4], Theorem 2).

According to [4], an ideal of a commutative semiprime ring S is said to be a *strongly Baer ideal* if, for any $x, y, z \in S$, $(x)^* \cap (y)^* = (z)^*$ and $x, y \in I$ imply $z \in I$. We now show that if I is a strongly Baer ideal of S, then for any $x \in S$ we have $x \in I$ if and only if $(x)^{**} \subseteq I$. Clearly, $x \in (x)^{**}$, and so if $(x)^{**} \subseteq I$, then $x \in I$. Suppose $x \in I$ and let $y \in (x)^{**}$. First we prove that $(xy)^* = (y)^*$. The inclusion $(y)^* \subseteq (xy)^*$ is obvious. If a lies in $(xy)^*$, i.e., $axy = 0$, then $ay \in (x)^*$. Since $y(x)^* = 0$, we get $ay^2 = 0$. Put $I = Say$. Then $I = 0$ because $I^2 = Sa^2y^2 = 0$ and S is semiprime. This gives $ay = 0$, whence $a \in (y)^*$. Thus we get $(xy)^* = (y)^*$. Therefore $(y)^* = (yx)^* \cap (0)^*$ (since $(0)^* = S$) and also $yx \in I$. Consequently, by the definition, $y \in I$, and hence $(x)^{**} \subseteq I$.

Thus for any $x \in S$ we have $x \in I$ if and only if $(x)^{**} \subseteq I$. In [4] it is proved that every strongly Baer ideal is the intersection of all prime strongly Baer ideals containing it (see [4], Lemma 4). Further, if S is an a.c. ring, then $B(R)$, defined as the set of all strongly Baer ideals of R, is a complete distributive lattice with $\cap$ as the infimum and, for any family $\{I_\alpha\} \subseteq B(R)$,

$$\bigvee I_\alpha = \{x \in S \mid (i_1)^* \cap \ldots \cap (i_n)^* \subseteq (x)^* \text{ for some } i_j \in I_{\alpha_j};\ j = 1, 2, \ldots, n\}$$

(see [4], Remark 2).

Throughout this paper, R denotes an a.c. ring with at least one nonzero divisor, and all ring ideals are assumed to be proper. Most of the topological concepts used in this paper are found in [5].

3. A topology for the prime strongly Baer ideals. Let

$$\gamma(R) = \{P \mid P \text{ is a prime strongly Baer ideal}\}$$

and, for each $x \in R$, let

$$\gamma(x) = \{P \in \gamma(R) \mid x \notin P\}.$$

THEOREM 1. *The class $\mu_R = \{\gamma(x) \mid x \in R\}$ forms a base for the open sets for a topology on $\gamma(R)$ and further $\gamma(R)$ is a T_0-space (with this topology), and the set of all compact open subsets of $\gamma(R)$ is μ_R. Also $\gamma(R)$ is compact.*

Proof. Clearly,

$$\gamma(x) \cap \gamma(y) = \gamma(xy) \quad \text{and} \quad \bigcup_{\gamma(x) \in \mu_R} \gamma(x) = \gamma(R),$$

so μ_R forms a base for the open sets for a topology on $\gamma(R)$. We show that each $\gamma(x)$ is compact in $\gamma(R)$. Let

$$\gamma(x) \subseteq \bigcup_{\alpha \in \Delta} \gamma(x_\alpha).$$

We claim that $x \in \bigvee (x_\alpha)^{**}$. Suppose $x \notin \bigvee (x_\alpha)^{**}$. Since $\bigvee (x_\alpha)^{**}$ is a strongly Baer ideal, it is the intersection of all prime strongly Baer ideals containing it. Therefore, there is a prime strongly Baer ideal P such that $x \notin P$ and $\bigvee (x_\alpha)^{**} \subseteq P$. Again since $x \notin P$, we have $P \in \gamma(x) \subseteq \bigcup \gamma(x_\alpha)$, and so $x_\alpha \notin P$ for some $\alpha \in \Delta$, which is a contradiction. Therefore $x \in \bigvee (x_\alpha)^{**}$. Now, as $x \in \bigvee (x_\alpha)^{**}$, there exist $i_1, i_2, \ldots, i_n \in R$ such that

$$i_j \in (x_{\alpha_j})^{**} \quad \text{and} \quad (i_1)^* \cap (i_2)^* \cap \ldots \cap (i_n)^* \subseteq (x)^*.$$

Observe that

$$\bigcap_{j=1}^{n} (x_{\alpha_j})^* \subseteq \bigcap_{j=1}^{n} (i_j)^* \subseteq (x)^*.$$

Since R is an a.c. ring, there exists $y \in R$ such that

$$(y)^* = \bigcap_{j=1}^{n} (x_{\alpha_j})^* \subseteq (x)^*.$$

Now we show that

$$\gamma(x) \subseteq \bigcup_{j=1}^{n} \gamma(x_{\alpha_j}).$$

Let $P \in \gamma(x)$. Then $x \notin P$. We claim that $x_{\alpha_j} \notin P$ for some $j \in \{1, 2, \ldots, n\}$. Suppose not. Then $x_{\alpha_j} \in P$ for $j = 1, 2, \ldots, n$. Since P is a strongly Baer ideal and

$$(y)^* = \bigcap_{j=1}^{n} (x_{\alpha_j})^*,$$

it follows that $y \in P$, and so $(x)^{**} \subseteq (y)^{**} \subseteq P$. Consequently, $x \in P$, a contradiction. Therefore $x_{\alpha_j} \notin P$ for some $j \in \{1, 2, \ldots, n\}$, and hence $P \in \gamma(x_{\alpha_j})$. This shows that

$$\gamma(x) \subseteq \bigcup_{j=1}^{n} \gamma(x_{\alpha_j}).$$

Thus, for each $x \in R$, $\gamma(x)$ is a compact open subset of $\gamma(R)$.

Let X be a compact open subset of $\gamma(R)$. Since X is open, we have

$$X = \bigcup_{\alpha \in \Delta} \gamma(x_\alpha).$$

As X is compact, it follows that

$$X = \bigcup_{i=1}^{n} \gamma(x_{\alpha_i}) \quad \text{for some } \alpha_1, \ldots, \alpha_n \in \Delta.$$

Again, since R is an a.c. ring, there exists $y \in R$ such that

$$(y)^* = \bigcap_{i=1}^{n} (x_{\alpha_i})^*.$$

We claim that $X = \gamma(y)$. Let $P \in X$. Then $x_{\alpha_i} \notin P$ for some $i \in \{1, 2, \ldots, n\}$. Since $(y)^* \subseteq (x_{\alpha_i})^*$, it follows that $(x_{\alpha_i})^{**} \subseteq (y)^{**}$, and so $y \notin P$. Consequently, $P \in \gamma(y)$, and therefore $X \subseteq \gamma(y)$. If $P \in \gamma(y)$, then $y \notin P$. As

$$(y)^* = \bigcap_{i=1}^{n} (x_{\alpha_i})^*$$

and P is a strongly Baer ideal by definition, $x_{\alpha_j} \notin P$ for some $j \in \{1, 2, \ldots, n\}$. Therefore $P \in \gamma(x_{\alpha_j}) \subseteq X$. Thus $X = \gamma(y)$, and hence μ_R is the set of all compact open subsets of $\gamma(R)$.

Next we prove that $\gamma(R)$ is a T_0-space. Suppose $P \neq Q$ for some $P, Q \in \gamma(R)$. Then either $P \nsubseteq Q$ or $Q \nsubseteq P$. Without loss of generality we assume that $P \nsubseteq Q$. Then there is an element $x \in P$ such that $x \notin Q$. So $P \notin \gamma(x)$ and $Q \in \gamma(x)$. Therefore $\gamma(R)$ is a T_0-space.

Finally, we prove that $\gamma(R)$ is compact. Suppose R has a nonzero divisor, say d. Then $d \notin P$ for every prime strongly Baer ideal P of R. So $\gamma(R) = \gamma(d)$, and hence $\gamma(R)$ is a compact space. This completes the proof of the theorem.

For any ideal I of R let

$$\gamma(I) = \{P \in \gamma(R) \mid I \nsubseteq P\}.$$

LEMMA 1. *Every open subset of $\gamma(R)$ is of the form $\gamma(I)$ for some strongly Baer ideal I of R.*

Proof. Let X be any open subset of $\gamma(R)$. Then

$$X = \bigcup_{\alpha \in \Delta} \gamma(x_\alpha).$$

So $X = \bigcup \gamma(x_\alpha) = \bigcup \gamma((x_\alpha)^{**}) = \gamma(\bigvee (x_\alpha)^{**})$, and $\bigvee (x_\alpha)^{**}$ is a strongly Baer ideal. Hence every open subset of $\gamma(R)$ is of the form $\gamma(I)$ for some strongly Baer ideal I of R.

THEOREM 2. *The lattice $B(R) = (B(R), \vee, \cap)$ is isomorphic to the lattice of all open subsets of $\gamma(R)$, and the mapping $I \to \gamma(I)$ $(I \in B(R))$ takes arbitrary lattice sums to corresponding set unions.*

Proof. By Lemma 1, every open subset of $\gamma(R)$ is of the form $\gamma(I)$ for some $I \in B(R)$, so that the map $I \to \gamma(I)$ from $B(R)$ to $\mathscr{S}$, the lattice of all open subsets of $\gamma(R)$, is onto. Also by Lemma 4 in [4] the above map is one-one. Again, for $\{I_\alpha\}_{\alpha \in \Delta} \subseteq B(R)$,

$$\gamma(\bigvee I_\alpha) = \bigcup \gamma(I_\alpha) \quad \text{and} \quad \gamma(I_\alpha \cap I_\beta) = \gamma(I_\alpha) \cap \gamma(I_\beta), \quad \alpha, \beta \in \Delta.$$

Therefore $B(R)$ is isomorphic to the lattice of all open subsets of $\gamma(R)$. This completes the proof of the theorem.

Let

$$\mathscr{M}(R) = \{P \mid P \text{ is a minimal prime ideal of } R\}$$

and for each $x \in R$ let

$$\mathscr{M}(x) = \{P \in \mathscr{M}(R) \mid x \notin P\}.$$

It is well known that the sets $\mathscr{M}(x)$ $(x \in R)$ form an open base for the open sets for a Hausdorff topology on $\mathscr{M}(R)$. We now give a characterization of quasiregular rings.

Integral domains and complete direct sums of integral domains are examples of quasiregular rings. For various characterizations of quasiregular rings the reader may refer to [1], [2] and [4].

THEOREM 3. *The following statements on R are equivalent:*

(i) *R is quasiregular.*

(ii) *For all $x, y \in R$ there is $x' \in (x)^*$ such that*

$$\mathcal{M}(y) \cap (\mathcal{M}(R) - \mathcal{M}(x)) = \mathcal{M}(yx').$$

(iii) $\gamma(R) = \mathcal{M}(R)$.

(iv) *Each $\gamma(x)$ is closed in $\gamma(R)$.*

(v) *$\gamma(R)$ is Hausdorff.*

(vi) *For every $y \in R$, $\mathcal{M}(y)$ is compact as a subset of $\mathcal{M}(R)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let $x, y \in R$. By (i) there is $x' \in R$ such that $(x)^{**} = (x')^*$. Observe that $xx' = 0$ and $(x+x')^* = \{0\}$. Let

$$P \in \mathcal{M}(y) \cap (\mathcal{M}(R) - \mathcal{M}(x)).$$

Then $y \notin P$ and $x \in P$. Since P is a minimal prime ideal and $(x+x')^* = \{0\}$, by Lemma 1.1 of [3] it follows that $x' \notin P$, and hence $P \in \mathcal{M}(yx')$. Therefore

$$\mathcal{M}(y) \cap (\mathcal{M}(R) - \mathcal{M}(x)) \subseteq \mathcal{M}(yx').$$

Now suppose $P \in \mathcal{M}(yx')$. Then $y \notin P$ and $x' \notin P$. As $xx' = 0$ and P is a prime ideal, we have $x \in P$, and so $P \in \mathcal{M}(y) \cap (\mathcal{M}(R) - \mathcal{M}(x))$. Thus

$$\mathcal{M}(y) \cap (\mathcal{M}(R) - \mathcal{M}(x)) = \mathcal{M}(yx'),$$

and hence (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Clearly, $\mathcal{M}(R) \subseteq \gamma(R)$. Let $P \in \gamma(R)$. Choose any $x \in P$ and $y \notin P$. By (ii), there is $x' \in (x)^*$ such that

$$\mathcal{M}(y) \cap (\mathcal{M}(R) - \mathcal{M}(x)) = \mathcal{M}(yx'),$$

so that $(y)^* = (yx + yx')^*$. Consequently, $x' \notin P$. Thus for every $x \in P$ there is $x' \notin P$ such that $xx' = 0$. Therefore, by Lemma 1.1 of [3], $P \in \mathcal{M}(R)$, and hence $\gamma(R) = \mathcal{M}(R)$.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds. Then for each $x \in R$ we have $\gamma(x) = \mathcal{M}(x)$, which is closed in $\mathcal{M}(R) = \gamma(R)$, and so (iv) holds.

(iv) $\Rightarrow$ (v). Suppose (iv) holds. Let P and Q be two distinct prime strongly Baer ideals of R. Choose $x \in P$ such that $x \notin Q$. Then $Q \in \gamma(x)$ and $P \in \gamma(R) - \gamma(x)$. Since $\gamma(x)$ is closed, there is $y \in R$ such that

$$P \in \gamma(y) \subseteq \gamma(R) - \gamma(x).$$

Also $\gamma(x) \cap \gamma(y) = \emptyset$. Thus (v) holds.

(v) $\Rightarrow$ (vi). Suppose (v) holds. Then $\gamma(R) = \mathcal{M}(R)$. Now the result follows from Theorem 1.

(vi) $\Rightarrow$ (i) follows from Theorem 3.4 in [3].

Remark. Theorems 1 and 3 show that R is quasiregular if and only if $\mathcal{M}(R)$ is a Boolean space. (A topological space X is called a *Boolean space* if X is compact, Hausdorff and has a base consisting of compact open subsets.)

We shall denote the closure, interior and exterior of a subset X of $\gamma(R)$ by $\mathrm{cl}X$, $\mathrm{int}X$ and $\mathrm{ext}X$, respectively. Clearly, for any subset X of $\gamma(R)$,

$$\mathrm{cl}X = \{P \in \gamma(R) \mid \bigcap_{q \in X} q \subseteq P\}.$$

We now prove some lemmas we need.

LEMMA 2. *If* $I \in B(R)$, *then* $I^* = \bigcap \{P \in \gamma(R) \mid I \nsubseteq P\}$.

Proof. Suppose $I \in B(R)$. Since R is semiprime, we have $I \cap I^* = \{0\}$, and so

$$I^* \subseteq \bigcap \{P \in \gamma(R) \mid I \nsubseteq P\}.$$

Let $x \notin I^*$. Then $xa \neq 0$ for some $a \in I$. Put $S = \{a, a^2, a^3, \ldots\}$. Obviously, S is a multiplicative subset of R (for the definition see [4]). Let

$$O(S) = \{b \in R \mid bc = 0 \text{ for some } c \in S\}.$$

By Lemma 6 and Definition 3 of [4], $O(S)$ is the intersection of all the minimal prime ideals containing it. Again, since $x \notin O(S)$, there is some minimal prime ideal P of R such that $x \notin P$ and $O(S) \subseteq P$. Clearly, by Lemma 1.1 in [3], $a \notin P$, and so $I \nsubseteq P$. Also P is a prime strongly Baer ideal. Therefore

$$x \notin \bigcap \{P \in \gamma(R) \mid I \nsubseteq P\}.$$

Consequently, $I^* = \bigcap \{P \in \gamma(R) \mid I \nsubseteq P\}$. This completes the proof of the lemma.

LEMMA 3. *Let* $I \in B(R)$. *Then*

(i) $\mathrm{cl}\gamma(I) = \gamma(R) - \gamma(I^*)$,

(ii) $\mathrm{int}(\gamma(R) - \gamma(I)) = \gamma(I^*)$, *and*

(iii) $I = I^{**}$ *if and only if* $\gamma(I) = \mathrm{int\,cl}\gamma(I)$.

Proof. (i) follows from Lemma 2.

(ii) $\mathrm{int}(\gamma(R) - \gamma(I)) = (\mathrm{cl}\gamma(I))' = (\gamma(R) - \gamma(I^*))' = \gamma(I^*)$, where $'$ denotes the set theoretic complementation.

(iii) follows from (i), (ii) and Theorem 2.

Using Lemma 2 we shall establish a necessary and sufficient condition for R to be quasiregular.

THEOREM 4. *R is quasiregular if and only if the exterior of every compact open subset of $\gamma(R)$ is compact.*

Proof. Suppose R is quasiregular. Let X be a compact open subset of $\gamma(R)$. Then, by Theorem 1, $X = \gamma(x)$ for some $x \in R$. Now

$$\mathrm{ext}X = \mathrm{ext}\gamma(x) = \mathrm{ext}\gamma(x)^{**} = \gamma((x)^*).$$

As R is quasiregular, $(x)^* = (y)^{**}$ for some $y \in R$. Therefore $\gamma((x)^*) = \gamma((y)^{**}) = \gamma(y)$, which is compact. Hence the exterior of every compact open subset of $\gamma(R)$ is compact.

Conversely, assume that the exterior of every compact open subset of $\gamma(R)$ is compact. Let $x \in R$. Then $\gamma(x)$ is a compact open subset of $\gamma(R)$. So $\operatorname{ext}\gamma(x) = \gamma(x^*)$ is compact and also open. Therefore, by Theorem 1, $\gamma((x)^*) = \gamma(y) = \gamma((y)^{**})$ for some $y \in R$, so that, by Theorem 2, $(x)^* = (y)^{**}$. This shows that R is quasiregular, which completes the proof of the theorem.

In [3] it is proved that $\mathcal{M}(R)$ is compact, Hausdorff and extremally disconnected if and only if for any ideal I in R there is $a \in R$ such that $I^* = (a)^*$ (see Theorem 4.4). The following theorem gives another equivalent condition for $\mathcal{M}(R)$ to be compact and extremally disconnected.

THEOREM 5. *The following two statements are equivalent*:

(i) *For any ideal I of R, $I^{**} = (a)^*$ for some $a \in R$.*

(ii) (a) *The exterior of every compact open subset of $\gamma(R)$ is compact, and*

(b) *the interior of the intersection of any family of compact open subsets of $\gamma(R)$ is compact.*

Proof. Suppose (i) holds. Then R is quasiregular, and so, by Theorem 4, the exterior of every compact open subset of $\gamma(R)$ is compact, therefore (a) holds.

Now we prove that (b) holds. Let $\{g_i \mid i \in \Delta\}$ be a family of compact open subsets of $\gamma(R)$. Then for each $i \in \Delta$ there is $x_i \in R$ such that $g_i = \gamma(x_i)$. Since R is quasiregular, $\gamma(R) = \mathcal{M}(R)$ and $\gamma(x_i) = \mathcal{M}(x_i)$ for every $i \in \Delta$. Also for every $x_i \in R$ there is $y_i \in R$ such that $(x_i)^* = (y_i)^{**}$. Observe that

$$\operatorname{int}\big(\bigcap_{i\in\Delta} g_i\big) = \operatorname{int}\big(\bigcap_{i\in\Delta} \mathcal{M}(x_i)\big) = \operatorname{int}\big(\mathcal{M}(R) - \bigcup_{i\in\Delta} \mathcal{M}(y_i)\big).$$

Let $I = \bigvee_{i\in\Delta} (y_i)^{**}$. Then

$$\operatorname{int}\big(\mathcal{M}(R) - \bigcup_{i\in\Delta} \mathcal{M}(y_i)\big) = \operatorname{int}\big(\mathcal{M}(R) - \mathcal{M}(I)\big) = \mathcal{M}(I^*) = \mathcal{M}\big((a)^{**}\big),$$

which, by hypothesis, equals $\mathcal{M}(a)$, a compact open subset of $\gamma(R)$. Therefore the interior of the intersection of any family of compact open subsets of $\gamma(R)$ is compact.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Then, by (a) and Theorem 4, R is quasiregular, and so, by Theorem 3, $\gamma(R) = \mathcal{M}(R)$. Let I be an ideal of R. Observe that

$$\mathcal{M}(I^{**}) = \operatorname{int}\operatorname{cl}\mathcal{M}(I^{**})$$

(by Lemma 3). Again notice that

$$\operatorname{cl}\mathcal{M}(I^{**}) = \mathcal{M}(R)-\mathcal{M}(I^{*})$$

$$= \mathcal{M}(R)-\mathcal{M}\Big(\bigcup_{x_\alpha\in I^*}(x_\alpha)^{**}\Big) = \bigcap_{x_\alpha\in I^*}\{\mathcal{M}(R)-\mathcal{M}(x_\alpha)^{**}\}.$$

Since R is quasiregular, for each $x_\alpha \in I^*$ there is $y_\alpha \in R$ such that $(x_\alpha)^* = (y_\alpha)^{**}$. Therefore

$$\bigcap_{x_\alpha\in I^*}\{\mathcal{M}(R)-\mathcal{M}(x_\alpha)^{**}\} = \bigcap_{\alpha}\{\mathcal{M}(y_\alpha)^{**}\}.$$

Thus

$$\mathcal{M}(I^{**}) = \operatorname{int}\Big(\bigcap_{\alpha}\{\mathcal{M}(y_\alpha)^{**}\}\Big)$$

and each $\mathcal{M}(y_\alpha)^{**}$ is a compact open subset of $\mathcal{M}(R)$, and so, by (b), $\mathcal{M}(I^{**})$ is a compact open subset of $\mathcal{M}(R)$. Consequently, there is $a\in R$ such that $\mathcal{M}(I^{**}) = \mathcal{M}(a) = \mathcal{M}((a)^{**})$, and hence $I^{**} = (a)^{**}$. This completes the proof of the theorem.

LEMMA 4. *Let* $\gamma_0(R) = \{P\in\gamma(R) \mid P = P^{**}\}$. *Then the subspace* $\gamma_0(R)$ *is discrete.*

Proof. Let $X = \{P_i \mid i\in\Delta\}$ be any subset of $\gamma_0(R)$. Then for each $i\in\Delta$ we have $P_i = (x_i)^*$ for some $0 \neq x_i \in P_i^*$. Suppose $P\in \operatorname{cl}X$. Then

$$\bigcap_{i\in\Delta}(x_i)^* \subseteq P.$$

If $x_i\in P$ for all $i\in\Delta$, then

$$P^* \subseteq \bigcap_{i\in\Delta}(x_i)^* \subseteq P,$$

and so $P^* = (0)$, which is absurd. So $x_i\notin P$ for some $i\in\Delta$. Consequently, $P = (x_i)^*$ for some $i\in\Delta$, and so $P\in X$. Therefore $X = \operatorname{cl}X$, and hence $\gamma_0(R)$ is discrete.

THEOREM 6. *The following statements are equivalent*:

(i) *R is quasiregular and every minimal prime ideal is nondense.*

(ii) *$\mathcal{M}(R)$ is compact and discrete.*

(iii) *$B(R)$ is a finite Boolean algebra.*

Proof. (i)$\Rightarrow$(ii) follows from Theorem 3 and Lemma 4.

(ii)$\Rightarrow$(iii). Suppose (ii) holds. Then $\gamma(R) = \mathcal{M}(R)$. Since $\gamma(R)$ is compact and discrete, the lattice of all open subsets of $\gamma(R)$ forms a finite Boolean algebra, and so, by Theorem 2, $B(R)$ is a finite Boolean algebra.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Clearly, every minimal prime ideal is nondense. Let $x \in R$. Then $(x)^* \vee (x)^{**} = R$. Choose any nonzero divisor $d \in ((x)^* \vee (x)^{**})$. Then

$$(i)^* \cap (j)^* \subseteq (d)^* = \{0\} \qquad \text{for some } i \in (x)^* \text{ and } j \in (x)^{**}.$$

Since $i \in (x)^*$, it follows that $(x)^{**} \subseteq (i)^*$. Now let $a \in (i)^*$ and $b \in (x)^*$. Then $ai = 0 = bj$; therefore

$$ab \in (i)^* \cap (j)^* \subseteq (d)^* = \{0\},$$

and hence $ab = 0$. Consequently, $(i)^* \subseteq (x)^{**}$, and so $(x)^{**} = (i)^*$. Therefore R is quasiregular. This completes the proof of the theorem.

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