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IRODUCTS AND SUMS OF ABSOLUTE PROPER RETRACTS, II

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In an earlier paper [2] the basic theorems regarding products and sums of absolute proper retracts (APR's) were established. In a remark at the end of that paper, it was stated that one of the theorems therein (Theorem 4.2) carried an unnecessary hypothesis. It was further stated that this situation would be later remedied, using some work on homotopy theoretical characterizations of APR's. This work was carried out in [3], and it is the purpose of this brief note to honor the earlier promise to remove the unneeded hypothesis in Theorem 4.2 of [2] by proving the following (Our notations and conventions are as in [2].)

THEOREM 4.2′. *Suppose that X_1 and X_2 are closed subsets of $X = X_1 \cup X_2$ and that X and $X_0 = X_1 \cap X_2$ are absolute proper retracts. Then X_1 and X_2 are absolute proper retracts.*

The homotopy theoretical characterization of APR's given by Theorem 4.5 of [3] is as follows:

THEOREM A. *Suppose that $X \in ANR$. Then $X \in APR$ if and only if X is non-compact, contractible, and docile at infinity.*

The notion of docility at infinity is given by Definition 3.17 of [3]. For our purposes it is only necessary to note that, for ANR's, docility at infinity is characterized by the following equivalent conditions (cf. Theorem 3.16 of [3]):

(i) X is *K-connected at infinity* (X is K-$\mathscr{C}(\mathscr{E})$) if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that if K_0 is a compact polyhedron and $f: K_0 \to X-B$ is a map carrying K_0 into a component of $X-B$, then f is nullhomotopic in $X-A$;

(ii) X is an *absolute extensor at infinity for compacta* ($X \in AEC(\mathscr{E})$) if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that if $D_0 \subset C_0$ are compacta and $f: D_0 \to X-B$ is a map carrying D_0 into a component of $X-B$, then there exists a map $f^*: C_0 \to X-A$ such that $f^*(d) = f(d)$ for all $d \in D_0$.

Proof of Theorem 4.2′. By symmetry we need only to show that $X_1 \in APR$. By Theorem 2.3 of [2] and Theorem 6.1 of Chapter IV of [1], $X_1 \in AR$. Since X_0 is closed in X_1 and $X_0 \in APR$, it also follows from Theorem A that X_1 is non-compact. Thus we need only to show that X_1 is docile at infinity. For this, we shall show that X_1 is K-$\mathscr{C}(\mathscr{E})$.

Let A be a compact subset of X_1. Then $A_0 = A \cap X_0$ is a compact subset of X_0 and, since $X_0 \in AEC(\mathscr{E})$, there exists a compact set $B_0 \subset X_0$ such that if $D_0 \subset C_0$ are compacta and $f\colon D_0 \to X_0 - B_0$ is a map carrying D_0 into a component of $X_0 - B_0$, then there exists a map $f^*\colon C_0 \to X_0 - A_0$ such that $f^*(d) = f(d)$ for all $d \in D_0$. Since X_0 is locally connected, there is no loss of generality in assuming that $X_0 - B_0$ has only finitely many components, say $Y_1, Y_2, \ldots, Y_m$. Let $A' = B_0 \cup A$. Then A' is a compact subset of X and, since X is K-$\mathscr{C}(\mathscr{E})$, there exists a compact set $B' \subset X$ such that if K_0 is a compact polyhedron and $f\colon K_0 \to X - B'$ is a map carrying K_0 into a component of $X - B'$, then f is nullhomotopic in $X - A'$. Let $B = B' \cap X_1$. We shall show that if K_0 is a compact polyhedron and $f\colon K_0 \to X_1 - B$ is a map carrying K_0 into a component of $X_1 - B$, then f is nullhomotopic in $X_1 - A$, thereby completing the proof.

Let K_0 be a compact polyhedron and let $f\colon K_0 \to X_1 - B$ be a map carrying K_0 into a component of $X_1 - B$. We may assume K_0 to be connected; for if this is not the case, we may — since components of $X_1 - B$ are path connected — replace K_0 by the union of K_0 and the cone over its 0-skeleton. Then $f(K_0) \subset X - B'$, and hence, by our choice of B', there exists a map $f_0\colon K \to X - A'$, where K is the cone over K_0, such that $f_0(x) = f(x)$ for all $x \in K_0$. (Here we follow the usual convention of regarding K_0 as a subset of K and identifying a nullhomotopy of a map on K_0 with an extension of the map to K.) Let P_0 be the component of $f_0^{-1}(X_1)$ containing K_0. Since K is locally connected and contractible, it follows from Corollary 9.3 of Chapter VII of [4] that each component of $K - P_0$ has a connected frontier. It follows that the frontier of each such component must be mapped by f_0 into one of the sets $Y_1, Y_2, \ldots, Y_m$. For $1 \leqslant i \leqslant m$, let P_i denote the closure of the union of those components of $K - P_0$ whose frontiers are mapped by f_0 into Y_i. Then $P_1, P_2, \ldots, P_m$ are pairwise disjoint, compact, and

$$K = \bigcup_{i=0}^{m} P_i.$$

Furthermore, f_0 carries $\mathrm{Fr} P_i$ into Y_i for $1 \leqslant i \leqslant m$. Thus, by our choice of B_0, for $i = 1, 2, \ldots, m$ there exists a map $f_i\colon P_i \to X_0 - A_0$ such that $f_i(x) = f_0(x)$ for all $x \in \mathrm{Fr} P_i$. Now, let $F\colon K \to X$ be defined by $F(x) = f_i(x)$ if $x \in P_i$ $(i = 0, 1, \ldots, m)$. We note that F is continuous, that $F(x) = f_0(x) = f(x)$ for all $x \in K_0$, and that $F(K) \subset X_1 - A$. Thus f is nullhomotopic in $X_1 - A$, completing the proof.

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