COLLOQUIUM MATHEMATICUM

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THREE RESULTS ON *I*-SETS

BY

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WITH BEST WISHES TO STANISLAW HARTMAN ON HIS 70TH BIRTHDAY

1. Closures of I-sets. Let Γ denote a discrete abelian group which is infinite. We use $b\Gamma$, the group dual to G_d, for the Bohr compactification of Γ, where G is the group dual to Γ and G_d is G with the discrete topology.

Definition. A subset E of Γ is said to be an *I-set* if every bounded function of E can be extended to $b\Gamma$ as a continuous function. Equivalently, $E \subseteq \Gamma$ is an I-set if every bounded function on E is the restriction to E of an almost periodic function Γ [2, p. 32].

The following Theorem is due to C. Ryll-Nardzewski [2] when $\Gamma = R$, the real numbers. It was generalized in [3] to arbitrary metrizable l.c.a. groups and is stated here for discrete abelian groups.

THEOREM 1. *If $E \subseteq \Gamma$ is an I-set, then E has no points of Γ as cluster points in $b\Gamma$. Consequently, the union of E with any finite subset of Γ is again an I-set.*

In this section we show that Theorem 1 does not characterize Γ as a subset of $b\Gamma$. By an "ineffective" construction we prove Theorem 2 below.

THEOREM 2. *There is an element $\phi \in b\Gamma \setminus \Gamma$ which is outside the closure in $b\Gamma$ of every I-subset of Γ.*

Proof. We use $\# S$ for the cardinality of a set S. The number of I-sets in Γ is at most $2^{(\#\Gamma)}$. In [4, p. 32, Theorem 5] Hartman and Ryll-Nardzewski show that Γ contains an I-set of the same cardinality as Γ. Since every subset of an I-set is an I-set, the number of I-sets in Γ is exactly $2^{(\#\Gamma)}$. We well-order the I-subsets of Γ as $\{E_\alpha | \alpha < \Lambda\}$ so $\# \{E_\alpha : \alpha < \beta < \Lambda\} < \Lambda$.

That $\# G = 2^{\#\Gamma}$ is well known [7]. (We thank Professor Ross for bringing [7] to our attention.) Here is a proof. Since G is compact and infinite, $\# G \geqslant 2^{\aleph_0}$. Let T be the circle $|z| = 1$ of the complex plane. Then $\# G \leqslant \#(T^\Gamma) = (\# T)^{\#\Gamma} = (2^{\aleph_0})^{\#\Gamma} = 2^{\aleph_0 \cdot \#\Gamma} = 2^{\#\Gamma}$. Thus, when Γ is countable, we have the equality $\# G = 2^{\#\Gamma}$. When Γ is uncountable, there is an independent subset of Γ of the same cardinality as Γ. Clearly $\# G \geqslant 2^{\#\Gamma}$, which establishes the equality $\# G = 2^{\#\Gamma}$.

ϕ shall be inductively defined on a subgroup of G and then arbitrarily extended to all of G. The subgroup shall be inductively produced as an increasing union of subgroups. Here is the construction. For each $\alpha < \Lambda$, we shall choose a finite set $K_\alpha \subseteq G$ and a character ϕ_α on G_d. We shall let H_α be the group generated by $\bigcup_{\gamma \leqslant \alpha} K_\gamma$. We impose the three conditions (i)–(iii) below.

(i) $\alpha < \beta$ implies $\phi_\beta|_{H_\alpha} = \phi_\alpha|_{H_\alpha}$.

For each finite set $F = \{f_1, \ldots, f_n\} \subseteq G$ and each $X \subseteq \Gamma$, we let

$$\langle X, F\rangle = \{(\langle y, f_1\rangle, \ldots, \langle y, f_n\rangle) |\ y \in X\}.$$

Let $K_\alpha = \{g_{\alpha,1}, \ldots, g_{\alpha,n_\alpha}\}$. Condition (ii) is this:

(ii) $\langle\{\phi_\alpha\}, K_\alpha\rangle \subsetneqq \langle E_\alpha, K_\alpha\rangle^-$ in T^{n_α}.

Note that (ii) implies $\phi_\alpha \notin \bar{E}_\alpha$.

We well-order Γ also, $\Gamma = \{\gamma_\alpha |\ \alpha < \#\Gamma\}$. Condition (iii) is this:

(iii) For each $\alpha < \#\Gamma$, there exists $g_\alpha \in H_\alpha$ such that

$$\phi_\alpha(g_\alpha) \neq \langle \gamma_\alpha, g_\alpha\rangle.$$

Note that (iii) implies $\phi_\alpha \neq \gamma_\alpha$.

Our final ϕ is defined to be ϕ_α on H_α and we extend ϕ arbitrarily off $\bigcup_{\alpha < \Lambda} H_\alpha$. Clearly $\phi \in b\Gamma \setminus \Gamma$ and $\phi \notin \bar{E}_\alpha$ for $\alpha < \Lambda$. It remains to be shown how ϕ_α may be found so that (i), (ii) and (iii) hold.

Suppose that we have successfully chosen $\{K_\alpha\}_{\alpha<\beta}$ and $\{\phi_\alpha\}_{\alpha<\beta}$, $\beta < \Lambda$. Let G_β be the group generated by $\bigcup_{\alpha<\beta} H_\alpha$ and define ϕ'_β to be ϕ_α on H_α (and elsewhere as consistent). Suppose that every extension of ϕ'_β to all of G gives us a character in $\bar{E}_\beta$. Thus $\phi'_\beta + (G_\beta)^\perp \subseteq \bar{E}_\beta$. Then $G_\beta^\perp$ would be both a Helson set and a group. Since $\#(G_\beta) < \Lambda$, there is a set $S \subseteq G \setminus G_\beta$ of elements independent of G_β (and each other) of cardinality Λ. Thus $\#(G_\beta^\perp) \geqslant 2^{\# S}$, that is, $G_\beta^\perp$ is infinite. That is a contradiction, because an infinite group may not be a Helson set. Thus, some extension of ϕ'_β is not in $\bar{E}_\beta$. There is therefore some finite set K'_β on which that extension, which we call ϕ_β, has property (ii). If $\beta < \#\Gamma$, we choose an element g_β of G independent of the group generated by $(\bigcup_{\alpha<\beta} K_\alpha) \cup K'_\beta$. On g_β we define ϕ_β so that $\phi_\beta \neq \gamma_\beta$. Let $K_\beta = K'_\beta \cup \{g_\beta\}$. (ii) and (i) still hold.

That completes the induction and ends the proof of Theorem 2.

Remark 1. It is evident that we may choose ϕ to be a member of any previously assigned basic open set in $b\Gamma$. Such a set has the form

$U(\lambda; g_1, \ldots, g_n; \varepsilon) = \{\Psi \in b\Gamma \mid |\Psi(g_i) - \langle \lambda, g_i \rangle| < \varepsilon, 1 \leqslant i \leqslant n\}$. To do that we merely specify $\phi(g_i) = \lambda(g_i)$ as the -1st step in the inductive construction.

Remark 2. There are $2^{(2^{\#\Gamma})}$ such ϕ. When choosing K_α, include as a last step $h_\alpha \in G$ independent of all members of K_γ, $\gamma < \alpha$, and of the other members of K_α. On h_α let ϕ_α take one of two distinct possible values.

PROBLEM (**P 1292**). Let $E \subseteq \Gamma$ such that E is not the finite union of I-sets. Does there exist $\phi \in \bar{E}$ such that $\phi \notin \bar{F}$ for all I-sets F?

2. ***I*-sets generate.** The following Proposition 3 is an analogue of results of Varopoulos, Körner and others: there exists a Helson subset E of the circle group T such that $E+E = T$. Whether the full analogue holds for I-sets we do not know. Proposition 3 is of course trivial in the case of Z.

PROPOSITION 3. *Let Γ be a discrete abelian group. Then Γ contains an I-set E such that E generates Γ.*

Proof. Let E_1 be a maximal $\frac{1}{3}$-Kronecker subset of Γ, that is, for every $f\colon E_1 \to T$, there exists $x \in G$ such that $|f(\gamma) - \langle x, \gamma \rangle| \leqslant \frac{1}{3}$ for all $\gamma \in E_1$. Non-trivial $\frac{1}{3}$-Kronecker subsets of Γ exist if Γ contains an infinite independent subset consisting of elements of order at least 19, or an element of infinite order. That maximal such sets E_1 exist follows from the compactness of G, an argument we leave to the reader. Clearly E_1 is an I-set. Let Λ be the subgroup of Γ that is generated by E_1. Then Γ/Λ is a group of bounded order, and therefore Γ/Λ contains an independent set F which generates it. Let $E_2 \subseteq \Gamma$ be a set of elements such that $\gamma \mapsto \gamma + \Lambda$ maps E_2 one-to-one onto F.

Then E_2 is an independent set and $\bar{E}_2 \cap \bar{E}_1 = \emptyset$, since $(\bar{E}_2 + \bar{\Lambda})/\bar{\Lambda} = \bar{F} \cap \{0\} = \emptyset$ in $b(\Gamma/\Lambda)$ by Ramsey [3]. (That follows from the independence of F and the functorial properties of passage to the Bohr compactification, precisely $b(\Gamma/\Lambda) \simeq (b\Gamma/\bar{\Lambda})$.) Therefore $E_1 \cup E_2$ is an I-set. Of course $E_1 \cup E_2$ must generate Γ, since $E_2 + \Lambda = F$ generates $\Gamma/\Lambda = \Gamma/Gp(E_1)$. That proves the Proposition.

PROBLEM (**P 1293**). S. Hartman [6, pp. 112–113] shows that there exist I-sets $E \subseteq Z$ such that $Gp(\bar{E}) = bZ$. Do there exist such I-sets in all non-compact abelian groups?

3. Γ **is a Borel subset of** $b\Gamma$.

THEOREM 4. *Let Γ be a locally compact abelian group. Then Γ is a Borel subset of its Bohr compactification $b\Gamma$.*

The key idea of the proof of Theorem 4 is contained in the next Lemma.

LEMMA 5. *Let Γ be a divisible discrete abelian group. Then Γ is a Borel subset of $b\Gamma$.*

Proof. Since Γ is divisible, Γ is a direct sum of copies of Q and groups $Z(p^\infty)$ [5, p. 165]. Let E be a maximal independent subset of Γ. Then for every $n \geqslant 1$, each $\gamma \in E$ has an nth root γ_n (we choose one), and, if $E_n = \{\gamma_n\colon \gamma \in E\}$, then E_n is also a maximal independent set and $\Gamma = \bigcup_{n=1}^{\infty} Gp(E_n)$. Thus, it will suffice to show that $Gp(E_n)$ is a Borel subset of $b\Gamma$, for each $n = 1, 2, \ldots$ We give the proof for the case $n = 1$.

$\bar{E}$ is the Stone–Čech compactification of E, so E is an open subset of the compact set $\bar{E}$. Let E_∞ denote the elements of E that have infinite order. Then E_∞ is an open subset of $\bar{E}$ also, and

$$Q = E \cup (-E_\infty) \cup \{0\}$$

is a Borel subset of $b\Gamma$.

Let $1Q = Q$, let $mQ = Q + (m-1)Q$ for $m > 1$ and let $Q^{(m)} = Q \times \ldots \times Q$ (m times). We shall show that mQ is a Borel subset of $b\Gamma$, for all $m \geqslant 2$. Since $\bigcup_{1}^{\infty} mQ = Gp(E)$, that will suffice. The closure of $Q^{(m)}$ is denoted by $\bar{Q}^{(m)}$.

Let $\bar{Q}_1^{(m)} = \bar{Q}^{(m)}/\sim$, where $\sim$ is the equivalence relation $x = (x_1, \ldots, x_m) \sim (y_1, \ldots, y_m) = y$ if y is obtained by permutation of the elements of x, and $\bar{Q}^{(m)} = \bar{Q} \times \ldots \times \bar{Q}$ (m times). We give $\bar{Q}_1^{(m)}$ the quotient space topology. Because permutations of coordinates induce homeomorphisms of $\bar{Q}^{(m)}$, the projection of $\bar{Q}^{(m)}$ onto $\bar{Q}^{(m)}$ is open (and continuous). Note that $Q^{(m)}$ is an open subset of $\bar{Q}^{(m)}$, so the image $Q_1^{(m)}$ of $Q^{(m)}$ in $\bar{Q}_1^{(m)}$ is an open set.

Let $P^{(m)} = \{(x_1, \ldots, x_m) \in \bar{Q}^{(m)}\colon \sum x_j \in (m-1)\bar{Q}\}$. Then $P^{(m)}$ is a closed subset of $\bar{Q}^{(m)}$ and the image, F, of $\bar{Q}^{(m)} \setminus P^{(m)}$ is open in $\bar{Q}_1^{(m)}$.

Of course, the mapping $(x_1, \ldots, x_m) \mapsto \phi(x_1, \ldots, x_m) = \sum_1^m x_j$ from $\bar{Q}^{(m)}$ to $m\bar{Q}$ factors through $\bar{Q}_1^{(m)}$; let ϕ_1 be the (continuous) mapping from $\bar{Q}_1^{(m)}$ to $m\bar{Q}$ that is induced by ϕ.

We claim that the restriction of ϕ_1 to F is one-to-one and onto $m\bar{Q} \setminus (m-1)\bar{Q}$. Checking that the range of $\phi_1|_F$ is $m\bar{Q} \setminus (m-1)\bar{Q}$ is routine. To see that $\phi_1|_F$ is one-to-one, apply Lemma 6 (below), which asserts that if $(x_1, \ldots, x_m), (y_1, \ldots, y_m) \in \bar{Q}^{(m)} \setminus P^{(m)}$ then $\sum x_j = \sum y_k$ if and only if the y's are a permutation of the x's. That implies at once that 1 is one-to-one.

We now claim that $\phi_1|_F$ is a homeomorphism. Since ϕ_1 is continuous, that amounts to showing that if $\sum_1^m x_j^{(\alpha)}$ converges to $\sum_1^m y_j$, then $(x_1^{(\alpha)}, \ldots, x_m^{(\alpha)})$ converges to an appropriate permutation of $(y_1, \ldots, y_m)$.

Of course $(x_1^{(\alpha)}, \ldots, x_m^{(\alpha)})$ has an accumulation point $(z_1, \ldots, z_m) \in \bar{Q}^{(m)}$. Of course, by the continuity of addition, $\sum z_j = \sum y_j$. Since $\sum z_j \in m\bar{Q} \setminus (m-1)\bar{Q}$, $(z_1, \ldots, z_m) \notin P^{(m)}$. Since ϕ_1 is one-to-one, $(z_1, \ldots, z_m)$ is a permutation of $(y_1, \ldots, y_m)$. Of course, that applies to every accumulation point of $\{(x_1^{(\alpha)}, \ldots, x_m^{(\alpha)})\}_\alpha$ in $\bar{Q}^{(m)}$. Therefore ϕ_1^{-1} is continuous.

Since ϕ_1 is a homeomorphism from F to $m\bar{Q}\setminus(m-1)\bar{Q}$, ϕ_1 preserves Borel sets. But the image of $Q^{(m)}\setminus P^{(m)}$ in F is open, so $\phi(Q^{(m)}\setminus P^{(m)}) = mQ\setminus(m-1)Q$ is a Borel set in Γ. Therefore $Gp(E) = Gp(Q) = \bigcup_{1}^{\infty} mQ$ is a Borel set. That ends the proof of Lemma 5.

LEMMA 6. *Let E be an independent subset of the abelian group G. Let E_∞ denote the set of elements of E that have infinite order and let $Q = E\cup(-E_\infty)\cup\{0\}$. If*

$$\{x_1, \ldots, x_m\} \subseteq Q,\ m \geqslant 1, \qquad \{y_1, \ldots, y_m\} \subseteq Q,\ \sum_{1}^{m} y_k \notin (m-1)Q$$

and $\sum x_j = \sum y_k$, then the y_k's are a permutation of the x_j's. (In the above $0Q = \{0\}$, $mQ = (m-1)Q+Q$ for $m \geqslant 1$.)

Proof. The lemma holds when $m = 1$. We shall induct on m.

We may assume that the conclusion holds for $1 \leqslant m' < m$. Suppose $\{x_1, \ldots, x_m\} \subseteq Q$, $\{y_1, \ldots, y_m\} \subseteq Q$ and $\sum x_j = \sum y_j$. We shall show that that implies one of the following from which the conclusion of the Lemma (for m) will follow.

a) For some j_0, k_0, $x_{j_0} = y_{k_0}$.

b) $\sum x_j \in (m-1)Q$.

c) All x_j, y_k have infinite order.

In case a), $\sum\{x_j\colon j \neq j_0\} = \sum\{y_k\colon k \neq k_0\}$, and the inductive hypothesis supplies the required conclusion.

We may therefore assume $x_j \neq y_k$ for all $1 \leqslant j, k \leqslant m$. Here is how case b) (excluded by the hypotheses, of course), might arise.

Suppose there exists j_0 such that for all k, $x_{j_0} \neq \pm y_k$. Then

$$\sum\{x_j\colon \pm x_j = x_{j_0}\} + \sum\{x_j\colon x_j \neq \pm x_{j_0}\} - \sum y_k = 0.$$

By the independence of E, $\sum\{x_j\colon x_j = \pm x_{j_0}\} = 0$. Therefore $\sum x_j \in (m-1)Q$.

We may therefore assume that for each j there exists a k such that $x_j = -y_k$. Such elements x_j cannot have order two (by Case a)). Since if $x_j \in E$ has finite order not equal to two, $-x_j \notin Q$, x_j must have infinite order. (That $x_j = 0$ is ruled out by Case b).) Thus all x_j have infinite order. The same reasoning shows all y_k have infinite order as well.

We may renumber the x_j's, y_k's so that $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ are distinct. The first sentence of the preceding paragraph shows $r = s$. If some $x_j = -x_l$ for $1 \leqslant j < l \leqslant m$, then $\sum_{1}^{m} x_j \in (m-1)Q$, so not only are $x_1, \ldots, x_r$ distinct, but $x_j \neq -x_l$ for all $1 \leqslant j, l \leqslant m$. Therefore

$$\sum_{1}^{m} x_j = \sum_{1}^{r} m_j x_j \quad \text{and} \quad \sum m_j = m.$$

Similarly, $\sum_1^m y_k = \sum_1^r n_k y_k$ and $\sum_1^r n_k = m$. Therefore

$$\sum_1^r m_j x_j = \sum_1^r n_k y_k.$$

By renumbering the y's again, we may assume $y_1 = -x_1, \ldots, y_r = -x_r$, so

$$\sum_1^r m_j x_j = -\sum_1^r n_k x_n.$$

Therefore $\sum_1^r (m_j + n_j) x_j = 0$.

Since the x_j are independent, $(m_j + n_j) x_j = 0$. Since the x_j have infinite order, $m_j + n_j = 0$. But the m_j and n_j are all > 0. That contradiction completes the proof of Lemma 6.

Proof of Theorem 4. We first consider the case that Γ is discrete. Then Γ can be embedded in a divisible group Λ [5, p. 167]. By Lemma 5, Λ is a Borel subset of $b\Lambda$. Straightforward functorial arguments show that $\Gamma = \Lambda \cap b\Gamma$ and, hence, that Γ is a Borel subset of $b\Gamma$.

Now suppose that $\Gamma = R^n \times \Gamma_1$ where $n \geqslant 0$ and Γ_1 has a compact open subgroup Λ. Then Γ_1/Λ is a Borel subset of $b(\Gamma_1/\Lambda)$. Therefore Γ_1 is a Borel subset of $b\Gamma_1$, since $b\Gamma_1/\Lambda = b(\Gamma_1/\Lambda)$. Also, R^n is a Borel subset of bR^n, since R^n is σ-compact. Therefore $R^n \times \Gamma_1$ is a Borel subset of $b(R^n \times \Gamma_1) = bR^n \times b\Gamma_1$. That ends the proof of Theorem 4.

Remarks. (i) Some years ago Y. Meyer pointed out that the proof of [1, 1.9.1] assumes that Γ is a Borel subset of $b\Gamma$ (that assumption is easily skirted), and that whether Γ was indeed a Borel subset of $b\Gamma$ appeared to be unknown. (Insertion of the phrase "on a σ-compact subset of" at the end of line 19, p. 33 of [1] eliminates the difficulty.)

(ii) If one only wishes to know that Γ is μ-measurable for all $\mu \in M(\Gamma)$, then it suffices to show that Γ is an analytic subset of $b\Gamma$. Since $Q - Q$ is analytic whenever Q is, one concludes that mQ is analytic for all $m \geqslant 1$, and hence that Γ is analytic in $b\Gamma$. The work in the proof of Lemma 5 was needed to show that Γ was *Borel*, since $X - X$ is not necessarily Borel, even if X is Borel.

(iii) Analysis of the proof of Theorem 4 shows that Γ is more than a Borel set: Γ is countable union of sets, each of which is the intersection of a compact set with an open set.

(iv) We thank Professor Hartman for calling [6] to our attention, reading preliminary versions of this note with much care, and for suggesting the problem cited given at the end of Section 2; it is included here with his kind permission.

(v) We are grateful to K. Stromberg and S. Saeki for pointing out gaps in the earlier proof of Lemma 5. Their stronger version of Theorem 4 will appear in [8].

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