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COMPOSITION OPERATORS ON THE BERGMAN SPACE

BY

DAVID M. BOYD (CHAPEL HILL, NORTH CAROLINA)

1. Introduction. Let φ be a non-constant analytic function mapping the open unit disk U into itself. If f is analytic in U, write $C_\varphi(f) = f\circ\varphi$. It has been shown by Ryff [6] that C_φ is a bounded operator on H^p, $0 < p < \infty$. Composition operators on H^p for $p \geqslant 1$ were studied in depth by Schwartz [8] and by Shapiro and Taylor [9]. We present here* a study of composition operators on the Bergman space A^2 which demonstrates the interesting interrelation between the operator-theoretic properties of C_φ and the geometric- and function-theoretic properties of the inducing function φ. The *Bergman space* A^2 is the Hilbert space of functions f analytic in U, satisfying

$$\|f\|^2 = \frac{1}{\pi}\iint_U |f(z)|^2 dA < \infty,$$

where A is the planar Lebesgue measure on the unit disk. Furthermore,

$$(f, g) = \frac{1}{\pi}\iint_U f(z)\overline{g(z)}\, dA$$

gives the form of the inner product. Of particular interest, the function

$$K_\zeta(z) = \frac{1}{(1-\bar{\zeta}z)^2} = \sum_{n=0}^{\infty}(n+1)(\bar{\zeta}z)^n$$

for fixed $\zeta \in U$ serves as the reproducing kernel for the space; that is, if $f \in A^2$ and $|\zeta| < 1$, then $f(\zeta) = (f, K_\zeta)$. As a consequence we note the following computation:

$$\|K_\zeta\|^2 = (K_\zeta, K_\zeta) = K_\zeta(\zeta) = \frac{1}{(1-|\zeta|^2)^2}.$$

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The functions $s_n(z) = \sqrt{n+1}z^n$ for $n = 0, 1, 2, \ldots$ serve as an orthonormal basis for the space A^2. For a more thorough discussion of A^2, see Hille [5], p. 325-338, or Bergman [1].

2. Boundedness and norm estimates. We first note that if C_φ is a bounded operator on A^2, then $\|C_\varphi\| \geqslant 1$, since any constant function has $\lambda = 1$ as an eigenvalue. Without much difficulty, we show that two particular classes of analytic functions $\varphi: U \to U$ will induce bounded operators on A^2. This leads to a study of the general case.

By the lemma of H. A. Schwarz, an analytic function $\varphi: U \to U$ with $\varphi(0) = 0$ satisfies the inequality $|\varphi(z)| \leqslant |z|$ for all $z \in U$, and $|\varphi'(0)| \leqslant 1$. We shall refer to such a function as a Schwarz function.

THEOREM 2.1. *If $\varphi: U \to U$ is a non-constant Schwarz function, then C_φ is a bounded operator on A^2, and $\|C_\varphi\| = 1$.*

Proof. For $f \in A^2$, Littlewood's Subordination Principle (see [4], p. 10) implies

$$\int_0^{2\pi} |(f\circ\varphi)(re^{it})|^2 dt \leqslant \int_0^{2\pi} |f(re^{it})|^2 dt.$$

The conclusion follows by integrating with respect to rdr.

THEOREM 2.2. *If $\varphi: U \to U$ is a non-constant conformal map such that*

$$d \equiv \inf_{|z|<1} |\varphi'(z)| > 0,$$

then C_φ is a bounded operator on A^2 and $1 \leqslant \|C_\varphi\| \leqslant 1/d$.

Proof. If $f \in A^2$, we use the fact that $|\varphi'(z)|^2$ is the Jacobian of the transformation φ to write

$$\frac{1}{\pi}\iint_U |(f\circ\varphi)(z)|^2 dA \leqslant (d^{-2})\frac{1}{\pi}\iint_U \left|f\big(\varphi(z)\big)\right|^2 |\varphi'(z)|^2 dA \leqslant (d^{-2})\,\|f\|^2.$$

THEOREM 2.3. *If $\varphi: U \to U$ is a non-constant analytic function, then C_φ is a bounded operator on A^2 and*

$$1 \leqslant \|C_\varphi\| \leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

Proof. Let $k(z) = \big(\varphi(0)-z\big)/\big(1-\overline{\varphi(0)}z\big)$, a conformal map of U onto itself with $k\big(\varphi(0)\big) = 0$, and note that k is its own composition inverse,

$$k^{-1} = k \quad \text{with} \inf_{|z|<1} |k'(z)| = \frac{1-|\varphi(0)|}{1+|\varphi(0)|} > 0.$$

Now write $\psi = k\circ\varphi$, so that $\psi\colon U \to U$ is a Schwarz function and $\varphi = k^{-1}\circ\psi$. Thus, for $f\in A^2$ by Theorems 2.1 and 2.2 we have

$$\|f\circ\varphi\| = \|f\circ k^{-1}\circ\psi\| \leqslant \|f\circ k^{-1}\| \leqslant \|f\| \frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

We next see that the reproducing kernel K_ζ enables us to obtain a better lower bound for the norm of a composition operator.

THEOREM 2.4. *If C_φ is a composition operator on A^2, then*

$$\sup_{z\in U} \frac{1-|z|^2}{1-|\varphi(z)|^2} \leqslant \|C_\varphi\|.$$

Proof. As an estimate on the growth of a function f in A^2 (see [5], p. 326) we have

$$|f(z)| \leqslant \|f\|(1-|z|^2)^{-1}. \tag{2.1}$$

For fixed ζ in U consider $K_{\varphi(\zeta)}(z)$ noting that

$$\|K_{\varphi(\zeta)}\|^2 = K_{\varphi(\zeta)}(\varphi(\zeta)) = \frac{1}{(1-|\varphi(\zeta)|^2)^2}. \tag{2.2}$$

From (2.1) applied to $K_{\varphi(\zeta)}\circ\varphi$ and (2.2) we find that

$$K_{\varphi(\zeta)}(\varphi(\zeta)) \leqslant \|C_\varphi\| \frac{1}{1-|\varphi(\zeta)|^2}\,\frac{1}{1-|\zeta|^2}.$$

Since ζ was chosen arbitrarily, the result follows.

COROLLARY 2.1. $\|C_\varphi\| = 1$ *if and only if φ is a Schwarz function.*

3. Compact operators. An operator T on a Banach space is called *compact* if it takes the unit ball into a set whose closure is compact. This is equivalent to the statement that the image under T of every bounded sequence has a convergent subsequence. We present first an equivalent condition for a composition operator to be compact. The proof differs from that of Schwartz for H^p [8] only by using (2.1) instead of the analogous inequality for H^p and is omitted.

THEOREM 3.1. *The operator C_φ is compact on A^2 if and only if, for every sequence $f_n \to f$ uniformly on compact subsets of U and bounded in norm, the sequence $C_\varphi f_n \to C_\varphi f$ strongly in A^2.*

An application of this theorem to the orthonormal system $s_n(z)$ yields a growth restriction on $\|\varphi^n\|$.

COROLLARY 3.1. *If C_φ is compact on A^2, then* $\|\varphi^n\| = o(n^{-1/2})$.

THEOREM 3.2. *If $\varphi\colon U \to U$ is a non-constant analytic function with*

$$\iint_U (1-|\varphi'(z)|^2)^{-2}\,dA < \infty,$$

then C_φ is a compact operator on A^2.

Proof. Let $\{f_n\}$ be a sequence with $\|f_n\| \leqslant 1$ for all n, and $f_n \to 0$ uniformly on compact subsets of U. It will suffice to show that $\|C_\varphi f_n\| \to 0$. Let $\varepsilon > 0$ and choose a compact subdisk K of U large enough that

$$\frac{1}{\pi}\iint\limits_{U-K}\left(1-|\varphi(z)|^2\right)^{-2}dA < \frac{\varepsilon}{2}.$$

Since φ maps compact sets into compact sets, it is possible to find an n_0 such that $\left|f_n(\varphi(z))\right|^2 < \varepsilon/2$ for $n > n_0$ and $z \in K$. Using these estimates and (2.1) applied to $U-K$, we have $\|C_\varphi f_n\|^2 < \varepsilon$ for $n > n_0$.

Later we shall see that this condition is equivalent to the condition that C_φ is a Hilbert-Schmidt operator on A^2.

COROLLARY 3.2. *If $\|\varphi\|_\infty < 1$, then C_φ is compact on A^2.*

THEOREM 3.3. *Suppose that φ maps U univalently onto $R \subset U$ and that the closure of R contains only one point w_0 on the unit circle. If*

$$\lim_{n\to\infty}|\varphi'(z_n)| = \infty, \quad \textit{whenever } \lim_{n\to\infty}\varphi(z_n) = w_0,$$

then C_φ is a compact operator on A^2.

Proof. Let $\{f_n\}$ be a sequence with $\|f_n\| \leqslant 1$ for all n, and $f_n \to 0$ uniformly on compact subsets of U. Let $\psi = \varphi^{-1}$ be the composition inverse of φ. By hypothesis, $\lim|\psi'(w)| = 0$, where $w \to w_0, w \in R$. For given $\varepsilon > 0$, choose $H = \{w\colon |w-w_0| < \delta\} \cap R$ so that $|\psi'(w)|^2 < \varepsilon/2$ for $w \in H$. The set $R-H$ is contained in a compact set in U, and we can find n_0 such that $|f_n(w)|^2 < \varepsilon/2$ for $n > n_0$ and for $w \in R-H$. Thus, for $n > n_0$,

$$\|C_\varphi f_n\|^2 = \frac{1}{\pi}\iint\limits_{R-H}|f_n(w)|^2|\psi'(w)|^2 dA(w) + \frac{1}{\pi}\iint\limits_{H}|f_n(w)|^2|\psi'(w)|^2 dA(w)$$

$$\leqslant \frac{\varepsilon}{2}\cdot\frac{1}{\pi}\iint\limits_{R}|\psi'(w)|^2 dA(w) + \frac{\varepsilon}{2}\|f_n\|^2 \leqslant \varepsilon.$$

Here we used $|\psi'(w)|^2$ as the Jacobian of the transformation ψ, and $\psi(R) \subset U$. By Theorem 3.1, the operator C_φ is compact on A^2.

A theorem of Carathéodory asserts that every conformal mapping of U onto the interior of a Jordan curve has a one-to-one continuous extension to the closed disk $\overline{U}$. This suggests a simpler case.

COROLLARY 3.3. *Suppose φ maps U univalently onto a region $R \subset U$ whose boundary is a Jordan curve touching the unit circle only at w_0. If*

$$\lim_{z\to z_0}|\varphi'(z)| = \infty \quad \textit{for } w_0 = \varphi(z_0),$$

then C_φ is compact on A^2.

An analytic function φ: $U \to U$ is said to have an *angular derivative at a point* z^* on the unit circle if there are a w^* also on the unit circle and a complex number c such that, for any triangle Δ whose interior lies wholely within U and has a vertex at z^* (see [2], p. 32), we have

$$\lim \frac{\varphi(z)-w^*}{z-z^*} = c \quad \text{as } z \to z^* \text{ with } z \in \Delta.$$

Such a function is angle preserving at z^* with respect to the radii, and the closure of $\varphi(U)$ must contain w^*. Thus $\varphi(U)$ touches the unit circle rather gently at w^*. Shapiro and Taylor [9] have shown that this prevents the operator C_φ from being compact on H^2. A similar result is valid for operators on A^2.

THEOREM 3.4. *If* φ: $U \to U$ *has the angular derivative at some point on the unit circle, then the operator* C_φ *is not compact on* A^2.

Proof. The conclusion will follow from the existence of a sequence $\{f_n\}$, bounded in norm, with $f_n \to 0$ uniformly on compact subsets of U and $\|C_\varphi f_n\|$ bounded away from zero. We may assume that $z^* = 1$ and $w^* = 1$ without loss of generality, so that

$$\sup_{0<x<1} \frac{|1-\varphi(x)|}{1-x} \equiv M < \infty. \tag{3.1}$$

For $f \in A^2$ and $r < 1$, integration of the Fejér-Riesz inequality for $f_r(z) = f(rz)$ yields

$$\int_0^1 \int_{-r}^{r} |f(x)|^2 \, dx\, dr \leqslant \frac{\pi}{2} \|f\|^2. \tag{3.2}$$

We define a family of functions by

$$f_a(z) = (1-a)^{1/2}(1-z)^{-(a+1)/2} \quad \text{for } 1/2 < a < 1.$$

Using an estimate found in Duren [4], p. 65, we see that

$$\|f_a\|^2 \leqslant C_a, \quad \text{where } C_a = \int_{-\infty}^{\infty} (1+2t^2\pi^{-2})^{-(a+1)/2} dt.$$

Since $1+2t^2\pi^{-2} > 1$ and $a > 1/2$, it follows that $C_a \leqslant C_{1/2} < \infty$. Hence $\{f_a\}$ is uniformly bounded in norm. Clearly, $f_a \to 0$ uniformly on compact subsets of U as $a \to 1$. We need only to show that $\|C_\varphi f_a\|$ is bounded away from zero.

First note that direct computation gives

$$\int_0^1 \int_0^r |f_a(x)|^2 \, dx\, dr = 1. \tag{3.3}$$

Now, from (3.1) and (3.2) we find that

$$\frac{\pi}{2}\|C_\varphi f_a\|^2 \geqslant M^{-(a+1)}\int_0^1\int_0^r |f_a(x)|^2\,dx\,dr.$$

Therefore, by (3.3),

$$\frac{\pi}{2}\|C_\varphi f_a\|^2 \geqslant M^{-(a+1)},$$

which is bounded away from zero. It follows that C_φ is not compact.

This proof required only the estimate (3.1); however, a result of Julia and Carathéodory ([2], p. 32) shows that this implies the existence of the angular derivative at $z^* = 1$.

COROLLARY 3.4. *If* $\|\varphi\|_\infty = 1$ *and* $\varphi' \in H^\infty$, *then the operator* C_φ *is not compact on* A^2.

Proof. Such a function φ has a Lipschitz continuous extension to the closed disk. If we assume that $\varphi(1) = 1$, then (3.1) holds, which is equivalent to φ having an angular derivative at $z^* = 1$ by the remark above.

As examples, we note that $\varphi_a(z) = 1-(1-z)^a$, $0 < a < 1$, induces a compact operator by Corollary 3.3 while $\psi_\beta(z) = (z+\beta-1)/\beta$, $\beta > 0$, induces a non-compact operator by Theorem 3.4. In both cases the range touches the unit circle at $z = 1$. It may seem noteworthy that φ_a takes values inside a polygon inscribed in the unit circle. However, an example in Section 5 will show that compact operators may be induced by functions $\varphi\colon U \to U$ which map onto a Jordan domain whose boundary touches the unit circle smoothly, i.e., with continuously turning tangent.

4. Hilbert-Schmidt operators. An operator T on an infinite-dimensional Hilbert space H is called a *Hilbert-Schmidt operator* if there exists an orthonormal basis $\{g_n\}$ such that $\sum \|Tg_n\|^2 < \infty$. The Hilbert-Schmidt operators form a two-sided ideal in the ring of bounded operators on H and are a proper subset of the compact operators. The following characterization for Hilbert-Schmidt operators on A^2 is similar to one derived for H^2 by Shapiro and Taylor [9]:

THEOREM 4.1. *The operator* C_φ *is Hilbert-Schmidt on* A^2 *if and only if*

$$\iint_U \big(1-|\varphi(z)|^2\big)^{-2}\,dA < \infty.$$

Proof. Using the orthonormal basis s_n, and interchanging summation with integration, we have

$$\sum_{n=0}^{\infty}\|C_\varphi s_n\|^2 = \frac{1}{\pi}\iint_U \sum_{n=0}^{\infty}(n+1)|\varphi|^{2n}\,dA = \frac{1}{\pi}\iint_U (1-|\varphi|^2)^{-2}\,dA.$$

The following result (see [9]) proves useful in studying ideals of composition operators:

THEOREM 4.2. *Let J be an ideal in the ring of bounded operators on A^2, and let φ and ψ be analytic functions mapping U into U, where φ is one--to-one. If $C_\varphi \in J$ and $\psi(U) \subset \varphi(U)$, then $C_\psi \in J$.*

THEOREM 4.3. *If φ takes U into a polygon inscribed in the unit circle, then C_φ is a Hilbert-Schmidt operator on A^2.*

Proof. By Theorem 4.2, it suffices to assume that φ maps U univalently onto the inscribed polygon P. The question reduces to the integrability of $(1-|\varphi(z)|)^{-2}$ over the pre-image under φ of a subset at any vertex. If we assume that the vertex is at $z = 1$ and that $\varphi(1) = 1$, a local mapping argument based on the reflection principle ([2], p. 104) shows that $1-\varphi(z) = (1-z)^a h(z)$, $0 < a < 1$, on a relative neighborhood Δ of 1, where h is analytic and bounded away from zero on Δ. We may assume that $|1-z| < 1$ on Δ. Thus

$$(4.1)\qquad |1-\varphi(z)|^{-2} \leqslant \delta^{-2}|1-z|^{-2a} \quad \text{for } z \in \Delta \text{ and some } \delta > 0.$$

By choosing $\beta = a+1$, so that $2a < \beta$ and $1 < \beta < 2$, and by using an estimate found in Duren [4], p. 65, we find that

$$\iint_\Delta |1-z|^{-2a} dA \leqslant \iint_U |1-z|^{-\beta} dA \leqslant C\int_0^1 (1-r)^{1-\beta} dr \leqslant C/2-\beta.$$

Together with (4.1), this implies that

$$\iint_\Delta |1-\varphi(z)|^{-2} dA < \infty.$$

Since values of φ are taken within the polygon P, there is a constant $M > 0$ such that $|1-\varphi(z)| \leqslant M(1-|\varphi(z)|)$ for $z \in \Delta$. This comparison completes the proof by showing that

$$\iint_\Delta (1-|\varphi(z)|)^{-2} dA < \infty.$$

5. A family of operators. Shapiro and Taylor [9] considered composition operators induced by a particular family of functions. We consider the operators induced by this same family for the space A^2 and make comparisons with the results for H^2.

Let $f_a(z) = z(-\log z)^a$ for $z \in \{z\colon \operatorname{Re} z > 0,\ |z| < 1\}$, where both the logarithm and the power function have branch line the negative real axis. The important properties of f_a ([9], p. 484-485) are the following:

For any $a > 0$, there exist an ε_a, $0 < \varepsilon_a < 1$, and also a set $H(\varepsilon_a) = \{z\colon z < \varepsilon_a,\ \operatorname{Re} z > 0\}$, such that

(i) the function f_a maps $H(\varepsilon_a)$ univalently onto a Jordan domain R contained in $|w-1|<1$, and the boundary of R touches the circle $|w-1|=1$ only at the origin;

(ii) the boundary of R touches the circle $|w-1|=1$ smoothly at the origin; i.e., in a neighborhood of the origin the tangent line turns continuously as the point of tangency moves along the curve;

(iii) f_a extends continuously to the closure of $H(\varepsilon_a)$.

Now, for a given $a>0$, let g_a be a one-to-one conformal map of U onto $H(\varepsilon_a)$ with $g_a(1)=0$, and write

$$\varphi_a(z) = 1-f_a\big(g_a(z)\big) \quad \text{for } z\in U. \tag{5.1}$$

Thus, $\varphi_a(1)=1$ and φ_a maps U univalently onto a Jordan domain in U whose boundary touches the unit circle smoothly at $z=1$.

THEOREM 5.1. *For any $a>0$, C_{φ_a} is a compact operator on A^2.*

Proof. By Corollary 3.3 it suffices to show that $\lim|\varphi_a'(z)|=\infty$ as $z\to 1$, $z\in U$. Since g_a can be extended conformally across the point $z=1$ by the reflection principle, $|g_a'(z)|$ is bounded away from 0 near $z=1$. Thus, for some $k>0$,

$$\lim_{z\to 1}|\varphi_a'(z)| \geqslant k\lim_{\zeta\to 0}|f_a'(\zeta)| \geqslant k\lim_{\zeta\to 0}(-\log\zeta - a)(-\log\zeta)^{a-1} = \infty.$$

THEOREM 5.2. *For $a>3/2$, C_{φ_a} is a Hilbert-Schmidt operator on A^2; for $a\leqslant 1$, C_{φ_a} is not Hilbert-Schmidt.*

Proof. For convenience we omit the subscript a. We determine whether C_φ is Hilbert-Schmidt by testing the convergence of the integral $\iint_U (1-|\varphi|)^{-2}dA$. Since $|\varphi|<1$ except near $z=1$ and since g and its inverse extend conformally to a neighborhood of the origin, the question reduces to testing the convergence of the integral $\iint\big(1-|1-f(z)|\big)^{-2}dA$ over some neighborhood of the origin intersected with $H(\varepsilon)$. Furthermore, $f(z)=\overline{f(\bar z)}$, so we can consider the set

$$\Delta = \{z:\ 0<|z|<\delta,\ 0<\arg z<\pi/2\},$$

where δ is chosen so small that, for $0<r<\delta$,

$$-\log r \geqslant 2a+\pi/2, \tag{5.2}$$

$$r(-\log r)^{a+1} \leqslant \min[2^{-a},\ 2^{-a-1}a/\pi]. \tag{5.3}$$

For notational purposes we write $z=re^{i\theta}$, $u=\operatorname{Re}f(z)$, $v=\operatorname{Im}f(z)$, and $\gamma=\arg(-\log z)$. Thus we can write

$$u = r|-\log z|^a\cos(\theta+a\gamma) \quad\text{and}\quad v = r|-\log z|^a\sin(\theta+a\gamma).$$

On the set Δ, we have $\theta>0$, so $\gamma<0$ and, by (5.2) and the inequality $\tan t\geqslant t$, for $0\leqslant t<\pi/2$, we obtain

$$0<\theta+a\gamma<\pi/2 \quad\text{and}\quad v=\operatorname{Im}f(z)>0.$$

Recall another useful inequality, i.e.

$$t \geqslant \sin t \geqslant 2t/\pi \quad \text{for } 0 \leqslant t \leqslant \pi/2. \tag{5.4}$$

By applying (5.4) first to $\theta + a\gamma$ and then to $(-\gamma)$, we can show

$$\cos(\theta + a\gamma) \geqslant \frac{2a}{\pi} \frac{\theta}{|-\log z|}. \tag{5.5}$$

By our initial requirement (5.2) we have the estimate

$$-\log r \leqslant |-\log z| \leqslant 2(-\log r). \tag{5.6}$$

From (5.5), (5.6) and (5.3) we can show that

$$\frac{u}{v^2} \geqslant \frac{2a}{\pi} \frac{1}{r \cdot 2^{a+1}(-\log r)^{a+1}} \geqslant 2.$$

This, together with (5.6) and (5.3), yields $2v^2 \leqslant u \leqslant 1$. Therefore,

$$|1-f|^2 \leqslant (1-u)^2 + u/2 \leqslant (1-u/4)^2,$$

which implies that

$$1 - |1-f| \geqslant 1 - (1-u/4) = u/4.$$

Since $1 - |1-f| \leqslant 1 - (1-u) = u$, the question of integrability reduces to considering the integral of u^{-2} over Δ, that is

$$\int_0^{\pi/2} \int_0^{\delta} \frac{1}{r(-\log r)^2 \cos^2(\theta + a\gamma)} \, dr \, d\theta.$$

First note that, for the interval $0 < \theta < \pi/4$, there is a c such that $1 > \cos(\theta + a\gamma) > c > 0$. Therefore, on this interval the integral exists if and only if $a > 1/2$.

We next consider $\pi/4 < \theta < \pi/2$. From (5.5) and (5.6) we see that $\cos(\theta + a\gamma) \geqslant a/4(-\log r)$; thus, the question reduces to the integral

$$\int_0^{\delta} \frac{1}{r(-\log r)^{2a-2}} \, dr,$$

which exists if $a > 3/2$. Thus the operator is Hilbert-Schmidt if $a > 3/2$.

On the other hand, by estimates (5.4) and (5.6) we find that

$$\cos(\theta + a\gamma) \leqslant \frac{\pi}{2} - \theta \left[1 - \frac{a\pi}{2(-\log r)}\right] \equiv S, \qquad \frac{dS}{dr} = \frac{a\pi}{2r(-\log r)^2}.$$

If we assume that $a \leqslant 1$, we know $(-\log r)^{2a} \leqslant (-\log r)^2$. Since the integral

$$\int_{\pi/4}^{\pi/2} \int_{\pi/2-\theta}^{\varepsilon} \frac{1}{S^2}\, dS\, d\theta$$

does not exist for $a \leqslant 1$, the operator is not Hilbert-Schmidt on A^2. This completes the proof.

COROLLARY 5.1. *There exist univalent maps φ such that C_φ is a Hilbert-Schmidt operator on A^2, yet φ takes U onto a Jordan domain whose boundary touches the unit circle smoothly.*

In their study of these same functions φ_a, Shapiro and Taylor determined that C_{φ_a} is a Hilbert-Schmidt operator on the space H^2 if and only if $a > 2$. Thus we have the following comparison:

COROLLARY 5.2. *There exist univalent maps φ such that C_φ is a Hilbert-Schmidt operator on A^2, yet is not Hilbert-Schmidt on H^2.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA

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