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WIENER'S TEST FOR THE BROWNIAN MOTION ON THE HEISENBERG GROUP

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We prove here Wiener's test formulated in the complete analogy with the one for standard Brownian motion on R^d ($d \geqslant 3$) (see [6], p. 257). Some other analogies with the group R^d are also indicated.

1. Let H_d be the *Heisenberg group* (of degree d), i.e. the nilpotent Lie group whose underlying manifold is $C^d \times R$ with coordinates $(z_1, \ldots, z_d, t) = (z, t)$ and whose group law is

$$(z, t)(z', t') = (z+z', t+t'+2\,\mathrm{Im}\, z\cdot z'), \quad \text{where } z\cdot z' = \sum_1^d z_j \bar{z}'_j.$$

We introduce the group $\{\delta_r\colon 0 < r < \infty\}$ of *dilations* on H_d defined by $\delta_r(z, t) = (rz, r^2 t)$ which satisfy the distributive law

$$\delta_r\big((z, t)(z', t')\big) = \big(\delta_r(z, t)\big)\big(\delta_r(z', t')\big),$$

and we define the *norm function* ϱ by

$$\varrho(z, t) = (|z|^4 + t^2)^{1/4}, \quad \text{where } |z|^2 = z\cdot z.$$

This function satisfies $\varrho\big(\delta_r(z, t)\big) = r\varrho(z, t)$ (see [2] and [3]).

Let $z = x+iy$. Then, $x_1, \ldots, x_d, y_1, \ldots, y_d, t$ are real coordinates on H_d. We set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

where $X_1, \ldots, X_d, Y_1, \ldots, Y_d$ generate the Lie algebra of H_d. We put

$$L = \sum_{j=1}^d (X_j^2 + Y_j^2),$$

L being a left-invariant second-order differential operator on H_d. According to the result of Folland [2], L is subelliptic (hence hypoelliptic) and $c_d \varrho^{-2d}$ is the fundamental solution for L with source at 0 (c_d is a suitable constant depending only on d).

We shall denote by $p_s(u)$ the fundamental solution for

$$\frac{\partial}{\partial s} = \frac{1}{2} L \quad (s \in (0, \infty), u \in H_d).$$

It is a non-negative function of class $C^\infty((0, \infty) \times H_d)$, tending to zero (for every fixed $s \in (0, \infty)$) as u tends to infinity, and

$$\int_{H_d} p_s(u)\, du = 1 \tag{1}$$

(du stands for the ordinary Lebesgue measure on $H_d \approx R^{2d+1}$). Since $\delta_{s*}L = s^2 L$, we have

$$p_s(u) = s^{-(d+1)} p_1\big(\delta_{s^{-1/2}}(u)\big)$$

(for facts concerning functions p_s see [4] and the references therein).

PROPOSITION 1. *We have*

$$\int_0^\infty p_s(u)\, ds = 2c_d \varrho^{-2d}(u), \quad u \in H_d.$$

Proof. (i) The function

$$g(u) \equiv \int_0^\infty p_s(u)\, ds$$

is locally integrable on H_d. For, let $K \subset H_d$ be compact; then for $0 < t < \infty$ we have

$$\int_K \int_0^\infty p_s(u)\, ds\, du = \int_K \int_0^t p_s(u)\, ds\, du + \int_K \int_t^\infty p_s(u)\, ds\, du$$
$$= \int_0^t \int_K p_s(u)\, du\, ds + \int_K \int_t^\infty s^{-(d+1)} p_1\big(\delta_{s^{-1/2}}(u)\big)\, ds\, du$$
$$\leqslant \int_0^t \int_{H_d} p_s(u)\, du\, ds + \int_K \int_t^\infty s^{-(d+1)} \|p_1\|_\infty\, ds\, du = t + |K| d^{-1} t^{-d} \|p_1\|_\infty < \infty.$$

(ii) Using a standard method (see, e.g., [10], p. 196) we infer that $2^{-1}g$ is a fundamental solution for L.

(iii) g is a *homogeneous function of degree* $-2d$ (see [3], p. 446), i.e. $g(\delta_r(u)) = r^{-2d} g(u)$ (the same is obviously true for $c_d \varrho^{-2d}$). Indeed, we have

$$g(\delta_r(u)) = \int_0^\infty p_s(\delta_r(u))\, ds = \int_0^\infty r^{-(2d+2)} p_{sr^{-2}}(u)\, ds$$
$$= \int_0^\infty r^{-(2d+2)} p_s(u) r^2\, ds = r^{-2d} g(u),$$

since

$$p_s(\delta_r(u)) = s^{-(2d+2)/2} p_1\big(\delta_{s^{-1/2}}(\delta_r(u))\big)$$
$$= s^{-(d+1)} p_1\big(\delta_{(r^{-2}s)^{-1/2}}(u)\big) = r^{-(2d+2)} p_{r^{-2}s}(u).$$

(iv) If we put $T \equiv 2^{-1}g - c_d\varrho^{-2d}$, then T is a (distribution) solution of $LT = 0$. Since L is hypoelliptic, we have $T \in C^\infty(H_d)$. Because of (iii),

$$T(u) = r^{2d}T(\delta_r(u)), \quad u \in H_d,\ r > 0.$$

Thus $T(u) = 0$ for $u \in H_d$.

2. Let γ be a left-invariant *Brownian motion* on H_d associated with $X_1, \ldots, X_d, Y_1, \ldots, Y_d$, i.e. the diffusion process on H_d with differential generator $\frac{1}{2}L$ (see [9]). Its transition probability density $p(s, u, v)$ (relative to the Haar measure on H_d which is the ordinary Lebesgue measure in our case) is equal to $p_s(u^{-1}v)$ for $u, v \in H_d$.

PROPOSITION 2. *γ is a transient diffusion, i.e. for every compact B in H_d we have*

$$\lim_{t\to\infty} \mathrm{P}_u[\gamma(s) \in B \text{ for some } s > t] = 0. \tag{2}$$

Note that the proof of Port and Stone ([8], p. 162, Proposition 5.1, and p. 145 and 146) works for this case as well as for the case of Brownian motion in R^d $(d \geqslant 3)$, so we shall restrict ourselves only to proving the following

LEMMA 1 (cf. [8], (5.12), p. 162). *For every compact set B there are a compact set K of positive measure and a constant $\eta > 0$ such that*

$$\inf_{0\leqslant s\leqslant 1} \inf_{b\in B} \mathrm{P}_b[\gamma(s) \in K] = \eta > 0.$$

Proof. (i) $p_1 \geqslant 0$ and (1) imply that there is a *ball* M (i.e. $M = \{u \in H_d: \varrho(u) \leqslant r\}$ for some $r > 0$) in H_d such that

$$\int_M p_1(u)\,du = \eta > 0.$$

(ii) For given compact subsets B and M of H_d there is a compact K such that, for every $b \in B$, $M \subset b^{-1}K$ (take K such that $B\cdot M \subset K$).

(iii) We have

$$\mathrm{P}_b[\gamma(s) \in K] = \int_K p_s(b^{-1}u)\,du = \int_K s^{-(d+1)} p_1\big(\delta_{s^{-1/2}}(b^{-1}u)\big)\,du$$
$$= \int_K s^{-(d+1)} p_1\big(\delta_{s^{-1/2}}(b^{-1})\,\delta_{s^{-1/2}}(u)\big)\,du$$
$$= \int_A s^{-(d+1)} p_1(v)\, s^{d+1}\,dv \geqslant \int_M p_1(v)\,dv = \eta,$$

where

$$A = \delta_{s^{-1/2}}(b^{-1})\delta_{s^{-1/2}}(K) = \delta_{s^{-1/2}}(b^{-1}K).$$

Thus, since M is δ-convex, we obtain

$$A \supset \bigcap_{r\geq 1}\delta_r(b^{-1}K) \supset \bigcap_{r\geq 1}\delta_r(M) \supset M, \quad r = s^{-1/2},\ 0 < s \leqslant 1.$$

For $s = 0$ we have $\mathrm{P}_b[\gamma(s) \in K] = 1$, since $B \subset K$.

3. Now, by *potential* of a measure μ ($\mu \geqslant 0$) we mean a function $G\mu$ on H_d constructed as follows:

$$(G\mu)(u) = \int_{H_d} g(u, v)\mu(dv) \quad \text{with } g(u, v) = 2c_d\varrho^{-2d}(u^{-1}v),\ u, v \in H_d.$$

Let $K \subset H_d$ be a compact set, and $M(K)$ all non-negative measures with supports in K whose potentials are bounded from above by 1. We call the *capacity* of K the number

$$C(K) = \sup\{\mu(K):\ \mu \in M(K)\}.$$

The following properties of the capacity function $C(\cdot)$ are immediate consequences of the properties of the function g.

PROPOSITION 3 (cf. [8], Theorem 6.4, p. 169).

$$C(uA) = C(A), \quad u \in H_d,$$
$$C\big(\delta_r(A)\big) = r^{2d}C(A), \quad r > 0,$$
$$C(A^{-1}) = C(A).$$

From the probabilistic potential theory of Hunt (see [1]) it follows, in view of Propositions 1 and 2, that there is a measure $\mu_K \in M(K)$ such that $C(K) = \mu_K(H_d)$ and we have

$$\mathrm{P}_u[m_K < +\infty] = \int_{H_d} g(u, v)\mu_K(dv)\ \ (\equiv p_K(u)), \tag{3}$$

where m_K is a hitting time of K (compact), $u \in H_d$.

We have the following version of

WIENER'S TEST (cf. [5], p. 128, [6], p. 257, and [7]). *For Brownian motion on H_d, $\mathrm{P}_e(Z) = 0$ or 1 according as $\sum_{n\geq 1} 2^{-n\cdot 2d}C(B_n)$ converges or diverges, where B is a closed set clustering to ∞, B_n is the intersection of B with the spherical shell $2^{n-1} \leqslant \varrho(u) \leqslant 2^n$, Z is the event that $\big(t:\ \gamma(t) \in B\big)$ clusters to $+\infty$, and e is the unit element in H_d.*

A proof that the convergence of the series implies $\mathrm{P}_e(Z) = 0$ in view of (2) and (3) follows exactly the lines indicated in [5] and [6].

In order to adapt the proof of the converse, the following elementary lemma and the corollary to it are useful.

LEMMA 2 (cf. [3], Lemma 8.9, p. 449). *The norm function ϱ is subadditive, that is*

$$\varrho(uv) \leqslant \varrho(u)+\varrho(v) \quad \textit{for all } u, v \in H_d.$$

Proof. We have

$$(4)\qquad [\varrho((z,t)(z',t'))]^4 = \Big|\sum_{j=1}^{d}(z_j+z_j')(\bar{z}_j+\bar{z}_j')\Big|^2 + \Big(t+t'+2\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)^2$$

$$= \Big(|z|^2+2\sum_{j=1}^{d}\operatorname{Re} z_j\bar{z}_j'+|z'|^2\Big)^2+\Big(t+t'+2\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)^2$$

$$= |z|^4+t^2+|z'|^4+t'^2+2|z|^2|z'|^2+2tt'+$$

$$+4\Big(\operatorname{Re}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)^2+4\Big(\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)^2+4\Big(\operatorname{Re}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)|z|^2+$$

$$+4\Big(\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)t+4\Big(\operatorname{Re}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)|z'|^2+4\Big(\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)t'.$$

Using the inequalities

$$(ab+cd)\leqslant(a^2+c^2)(b^2+d^2)^{1/2},\qquad a,b,c,d\in R,$$

$$|z\cdot z'|\leqslant|z|\,|z'|\leqslant\varrho(z,t)\varrho(z',t'),$$

we get

$$|z|^2|z'|^2+tt'\leqslant\varrho(z,t)^2\varrho(z',t')^2,$$

$$\Big(\operatorname{Re}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)^2+\Big(\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)^2\leqslant\varrho(z,t)^2\varrho(z',t')^2,$$

$$\Big(\operatorname{Re}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)|z|^2+\Big(\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)t\leqslant\varrho(z,t)^3\varrho(z',t'),$$

$$\Big(\operatorname{Re}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)|z'|^2+\Big(\operatorname{Im}\sum_{j=1}^{d} z_j\bar{z}_j'\Big)t'\leqslant\varrho(z,t)\varrho(z',t')^3.$$

Substituting these inequalities to equation (4) we obtain the lemma.

COROLLARY. *For every $u, v \in H_d$ we have*

$$|\varrho(u)-\varrho(v)|\leqslant\varrho(u^{-1}v).$$

Proof of the converse. If the series is divergent, then

$$\sum_{n\geqslant1}2^{-(4n+j)2d}C(B_{4n+j}) = +\infty \quad \text{for some } j=0,1,2,3.$$

Suppose that the series with $j = 2$ is divergent. Let

$$m_n = \min\{t\colon \varrho(\gamma(t)) = 2^{4n}\}$$

be the crossing time (note that (2) implies $m_n < \infty$), and let $l_n = \gamma(m_n)$ be the crossing place ($n \geqslant 1$). Then we have (cf. [6], (3), p. 256, and [5], p. 129)

$$\begin{aligned} P_e[\gamma(t) \in B_{4n+2} \text{ for some } t \in [m_n, m_{n+1}) \mid B_{m_n}] \\ &\geqslant 2c_d \int_{B_{4n+2}} [\varrho(l_n^{-1}b)^{-2d} - E_{l_n}(\varrho(l_{n+1}^{-1}b)^{-2d})]\mu_{B_{4n+2}}(db) \\ &\geqslant 2c_d C(B_{4n+2})[(2^{4n}+2^{4n+2})^{-2d} - (2^{4(n+1)}-2^{4n+2})^{-2d}] \\ &= 2c_d C(B_{4n+2})2^{-(4n+2)2d}[(4/5)^{2d} - (1/3)^{2d}] \equiv Q_n \end{aligned}$$

and, consequently,

$$\begin{aligned} d_{n,m} &\equiv P_e[\gamma(t) \notin B_{4j+2},\, t \in [m_j, m_{j+1}),\, n \leqslant j \leqslant m] \\ &= E_e\big[P_e[\gamma(t) \notin B_{4m+2},\, t \in [m_m, m_{m+1}) \mid B_{m_m}],\, \gamma(t) \notin B_{4j+2}, \\ &\qquad\qquad t \in [m_j, m_{j+1}),\, n \leqslant j \leqslant m-1\big] \\ &\leqslant (1-Q_m)d_{n,m-1} \leqslant (1-Q_m)\ldots(1-Q_{n+1}). \end{aligned}$$

Since

$$\sum_{n\geqslant 1} Q_n = +\infty,$$

we get

$$P_e[\gamma(t) \notin B_{4j+2},\, t \in [m_j, m_{j+1}),\, j \geqslant n] = 0,$$

whence

$$P_e[\gamma(t) \in B,\, t \geqslant m_n] = 1 \quad \text{for every } n \geqslant 1$$

and $m_n \uparrow \infty$ as $n \uparrow \infty$ because of transience of γ. Finally, we notice that nothing essential changes if the series with $j = 0, 1$ or 3, at the beginning of the proof, is divergent.

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