COLLOQUIUM MATHEMATICUM

VOL. XLVI 1982 FASC. 2

MODULES WHOSE COUNTABLY GENERATED SUBMODULES ARE EPIMORPHIC IMAGES

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Cukerman [2], Shanny [5], and Ware [7] have shown that a ring R is Artin semisimple if and only if R has a nonfinitely generated free module with regular endomorphism ring. This answers a question raised in [6]. In an easy way we prove a stronger result and as a corollary we get the afore-mentioned theorem. Artinian projective modules need not be Noetherian (see [3]) but we give a class of projective modules for which this is true. Zelmanowitz [8] calls a module *regular* if every cyclic submodule is a projective direct summand. We observe some properties of modules whose cyclic submodules are merely direct summands.

Throughout this paper all rings are associative with a unity and modules are unitary right modules. A module has a *finite uniform dimension* if it contains no infinite direct sum of submodules. A ring R is called *regular* if given $a \in R$ there exists $a' \in R$ such that $a = aa'a$, and R is called *right self-injective* if it is injective as a right R-module. An R-module M is called *quasi-injective* if, given any R-submodule N of M and a homomorphism $f: N \to M$, f can be extended to a homomorphism $f: M \to M$. It is well known that if M is a quasi-injective R-module and $H = \mathrm{End}_R(M)$, then $H/J(H)$ is a regular, right self-injective ring, where $J(H)$ denotes the Jacobson radical of H.

Definition 1. An R-module M is called *regular* if every cyclic R-submodule of M is a direct summand.

Projective modules over regular rings, every right ideal in a regular ring, and semisimple modules are some examples of regular modules.

The following two lemmas are stated in [8].

LEMMA 1. *If M is a regular R-module, then every finitely generated R-submodule of M is a direct summand.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of a submodule N of M. Proceed by induction on n. For $n = 1$ the assertion follows from the definition. Let us assume that it is also true for submodules generated by less than n elements. Since $x_n R$ is a direct summand of M, there exists a homomorphism θ_n: $M \to N$ such that $\theta_n(x_n) = x_n$. Let us set $y_i = x_i - \theta_n(x_i)$, $i = 1, \ldots, n$. We note that $y_n = 0$. Then by the induction hypothesis there exists a homomorphism θ': $M \to N$ such that $\theta'(y_i) = y_i$, $i = 1, \ldots, n$. Now define θ: $M \to N$ by $\theta = \theta_n + \theta'(1-\theta_n)$. Clearly, $\theta(x_i) = x_i$ and, therefore, the sequence $0 \to N \to M$ splits.

LEMMA 2. *Every countably generated regular R-module is a direct sum of cyclic submodules.*

The proof goes by an easy induction (see [8]).

COROLLARY 1. *If M is a regular module with a finite uniform dimension, then M is a finite direct sum of simple submodules.*

Proof. We note that every submodule is regular. Therefore, by Lemma 2, every submodule must be finitely generated, and hence is a direct summand. Thus M is semisimple, and since it has a finite uniform dimension, the proof is completed.

The following gives a converse to Schur's lemma.

PROPOSITION 1. *If M is a regular R-module such that $H = \mathrm{End}_R(M)$ is a division ring, then M must be a simple R-module.*

Proof. It is sufficient to show that $mR = M$ for each $m \in M$ $(m \neq 0)$. Now take $m \in M$ $(m \neq 0)$. Then, since M is regular, there is a projection $p: M \to mR$. Therefore, if i: $mR \to M$ is a natural inclusion, then $0 \neq ip \in H$. By the assumption on H, ip is an isomorphism, whence $M = \mathrm{Im}(ip) = mR$, which completes the proof.

PROPOSITION 2. *If every injective R-module is regular, then R is Artin semisimple.*

Proof. We note that every cyclic R-module is injective as a direct summand of its injective envelope. Now apply Osofsky's result [4].

We also observe the following properties of regular modules.

Remark 1. *If M is a regular R-module, then $J(M) = 0$.*

To see this we show that given any $x \in M$ $(x \neq 0)$ there exists a maximal submodule not containing x. Since M is regular, we have $M = xR \oplus P$ for some submodule P. Now, an easy application of Zorn's lemma shows that there exists a submodule N of M maximal with respect to the property that $N \supseteq P$, $x \notin N$. But, clearly, N is a maximal submodule. We also observe that if M is a regular faithful R-module, then $MJ(R) \subseteq J(M)$ implies $J(R) = 0$. Hence the annihilator of M in R is an intersection of maximal right ideals.

Next we characterize modules whose countably generated submodules are direct summands.

LEMMA 3. *If every countably generated submodule of an R-module M is finitely generated, then M is Noetherian.*

Proof. It is sufficient to show that M has acc on finitely generated submodules. Let $N_1 \subset \ldots \subset N_n \subset \ldots$ be an infinite ascending chain of finitely generated submodules. Then $\bigcup_{i=1}^{\infty} N_i$ is a countably generated submodule which must be finitely generated, so the chain terminates.

COROLLARY 2. *If M is an R-module whose countably generated submodules are direct summands, then M is semisimple.*

Proof. Let N be a finitely generated submodule of M. Clearly, every countably generated submodule of N is a direct summand of N. Now, by Lemma 3, N is Noetherian. Since N is also regular, it is semisimple. But $M = \sum_{x \in M} xR$ shows that M is a sum of simple submodules, i.e. M is semisimple.

Definition 2. An R-module M is said to be a *C-module* if, given any countably generated submodule N, there exists an epimorphism $f\colon M \to N$.

Infinitely generated free R-modules and semisimple modules are C-modules. There exists also a projective C-module which is neither free nor semisimple. Indeed, let F be a nonfinitely generated free R-module and suppose R' is a ring not isomorphic to R. Put $A = R \times R'$. Then F becomes in a natural way an A-module, and it is clear that F is a projective C-module over A.

It is well known that an Artinian projective module cannot be Noetherian (see [3]). But we have the following

PROPOSITION 3. *Artinian projective C-modules are Noetherian.*

Proof. Suppose M is an Artinian projective C-module. It is well known that an Artinian projective module is finitely generated (see [3]). Now Lemma 3 completes the proof.

Definition 3. Let M be an R-module and $H = \mathrm{End}_R(M)$. Then

$$A = \{f \in H\colon \mathrm{Im} f \text{ is contained in a countably generated submodule of } M\}$$

is an ideal of H called the *ideal of endomorphisms of countable rank.*

THEOREM. *If M is a C-module whose ideal of endomorphisms of countable rank is a regular ring, then M is semisimple.*

Proof. Let N be a countably generated submodule of M. Then there exists an epimorphism $M \xrightarrow{f} N \to 0$. Now, there exists $g\colon M \to M$ such that $f = fgf$, so $fg\colon M \to N$ splits the sequence $0 \to N \to M$. Thus every countably generated submodule of M is a direct summand and the Theorem follows by Corollary 2.

Remark 2. Bass [1] has shown that infinitely generated projective modules tend to be free. This roughly means that, as a projective module becomes greater, free summands tend to appear.

COROLLARY 3. *Let M be a C-module over a ring R. If the ideal of endomorphisms of M of countable rank is a regular ring and M has a nontrivial free direct summand, then R is Artin semisimple.*

Definition 4. An R-module M is called *nonsingular* if

$$Z(M) = \{X \subset M\colon \mathrm{Ann}(X) \text{ is an essential right ideal}\} = 0.$$

COROLLARY 4. *If P is a nonsingular C-module over R such that $H = \mathrm{End}_R(P)$ is a right self-injective ring, then P is semisimple.*

Proof. Since the endomorphism ring of a nonsingular quasi-injective module is a self-injective regular ring, it is sufficient to show that P is quasi-injective. So let N be a submodule of P and suppose $f\colon N \to P$ is a homomorphism. We must show that f can be extended to an element $f' \in H$. Put $A = \mathrm{Hom}_R(P, N)$. Then A is a right ideal of H. Now we consider a homomorphism $\theta\colon A \to H$ defined by $\theta(a) = fa$ for all $a \in A$. Since H is a right self-injective ring, θ can be extended to $\theta'\colon H \to H$. Therefore, if we put $\theta'(1) = f'$, then $\theta'(a) = f'a = fa$ for all $a \in A$. We now claim that f' extends to f. Indeed, if $0 \neq x$ is any element of N, then there exists $h \in A$ such that $h(P) = xR$. Therefore, there exists $m \in P$ such that $h(m) = x$. Now $f'(x) = f'\big(h(m)\big) = f'h(m) = fh(m) = f(x)$.

COROLLARY 5. *If R is a nonsingular ring such that the ring of infinite row matrices over R is right self-injective, then R must be Artin semisimple.*

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Reçu par la Rédaction le 22. 1. 1979;
en version modifiée le 23. 11. 1979