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ON AUTOMORPHISM GROUPS OF RELATIONAL SYSTEMS AND UNIVERSAL ALGEBRAS

BY

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Let X be a non-empty set and a an arbitrary non-zero ordinal. We denote by X^a the a-th power of X, i.e. $X^a = X \times \ldots \times X$ (a times). Any subset of X^a we call an *a-ary relation* in X and any mapping $f: X^a \to X$ we call an *a-ary operation* in X.

A mapping $\varphi: X \to X$ will be called a *permutation* of X if φ is 1-1 and onto. We accept the notation φx instead of $\varphi(x)$ to denote the value of φ for an argument x. A permutation φ of X will be called an *automorphism* of the relation $r \subseteq X^a$ if for any sequence $(a_0, \ldots, a_\beta, \ldots)_{\beta<a}$ of elements of X we have $r(a_0, \ldots, a_\beta, \ldots)_{\beta<a}$ iff $r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<a}$.

The group of all automorphisms of the relation r in X we denote by $\mathrm{Aut}(X; r)$. Analogously, a permutation φ will be called an automorphism of the operation $f: X^a \to X$ if for any sequence $(a_0, \ldots, a_\beta, \ldots)_{\beta<a}$ of elements of X we have $\varphi f(a_0, \ldots, a_\beta, \ldots)_{\beta<a} = f(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<a}$. The group of all automorphisms of the operation f in X is denoted by $\mathrm{Aut}(X; f)$.

A *relational system* is a pair $(X; \boldsymbol{R})$, where X is a non-empty set and $\boldsymbol{R}$ is a family of relations in X. An *algebra* is a pair $(X; \boldsymbol{F})$, where X is a non-empty set and $\boldsymbol{F}$ is a family of operations in X. We put

$$\mathrm{Aut}(X; \boldsymbol{R}) = \bigcap_{r \in \boldsymbol{R}} \mathrm{Aut}(X; r) \quad \text{and} \quad \mathrm{Aut}(X; \boldsymbol{F}) = \bigcap_{f \in \boldsymbol{F}} \mathrm{Aut}(X; f).$$

By the *partition* π of a non-empty set X we mean the family of sets $\{X_i\}_{i \in I}$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_{i \in I} X_i = X$. We admit the case where some X_i is empty.

In this paper we prove (Theorem 1) that for every a-ary relation r in a non-empty set X it is possible to define two operations $f_r: X^a \to X$ and $g_r: X^2 \to X$ such that $\mathrm{Aut}(X; r) = \mathrm{Aut}(X; f_r, g_r)$, which implies interesting corollaries for relational systems and algebras in Section 2.

1. Let X be a non-empty set and $\pi = \{X_i\}_{i\in I}$ be a partition of X. If we accept the axiom of choice when $|I| > \aleph_0$, then we have the following

LEMMA. *There exists a binary operation* $g: X^2 \to X$ *such that* $\varphi \in \mathrm{Aut}(X; g)$ *iff, for every* $i \in I, \varphi(X_i) = X_i$.

Proof. From the axiom of choice it follows that I can be considered as a set of ordinals and, therefore, we can put the sets $X_i, i \in I$, into the sequence $(X_0, \ldots, X_\beta, \ldots)_{\beta<\varkappa}$, where $\varkappa = \min\{\gamma: \bar{\gamma} = |I|\}$.

Let us define the operation $g: X^2 \to X$ putting

$$g(a, b) = \begin{cases} a & \text{if } a, b \in X_\beta \text{ for some } \beta < \varkappa, \\ \{a, b\} \cap X_{\min\{\beta_1,\beta_2\}} & \text{if } a \in X_{\beta_1}, b \in X_{\beta_2}, \beta_1 \neq \beta_2. \end{cases}$$

The sufficiency is trivial.

We prove the necessity. If $\varkappa = 1$, then the proof is trivial. If $\varkappa > 1$, we use induction with respect to β. Put $\beta = 0$ and assume that, for some $a \in X_0, \varphi a \in X - X_0$. Then $\varphi^{-1}a \neq a$ and by the definition of g we get $\varphi g(a, \varphi^{-1}a) = \varphi a$ and $g(\varphi a, \varphi\varphi^{-1}a) = a$. Thus φ is not an automorphism of $(X; g)$. If $a \in X - X_0$ and $\varphi a \in X_0$, then φ is not an automorphism because otherwise φ^{-1} would be an automorphism which contradicts the first part of the proof.

Assume now that for any $\gamma < \beta < \varkappa$ we have $\varphi(X_\gamma) = X_\gamma$. If for some $a \in X_\beta$ we have $\varphi a \notin X_\beta$, then, by the induction hypothesis, $\varphi a, \varphi^{-1}a \notin X_\gamma$ for every $\gamma < \beta$ and, therefore,

$$\varphi a = \varphi g(a, \varphi^{-1}a) = g(\varphi a, \varphi\varphi^{-1}a) = g(\varphi a, a) = a$$

which is a contradiction. Thus we have proved that $\varphi(X_\beta) \subseteq X_\beta$.

Since the mapping φ^{-1} also is an automorphism, the same argument shows that $\varphi^{-1}(X_\beta) \subseteq X_\beta$ which gives finally that $\varphi(X_\beta) = X_\beta$ for any $\beta < \varkappa$, q.e.d.

THEOREM 1. *For every a-ary relation r in a non-empty set X it is possible to define two operations:* $f_r: X^a \to X$ *and* $g_r: X^2 \to X$ *such that* $\mathrm{Aut}(X; r) = \mathrm{Aut}(X; f_r, g_r)$.

Proof. Let us define the operation f_r by putting, for every $(a_0, \ldots, a_\beta, \ldots)_{\beta<a} \in X^a$,

(i) $f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<a} = a_0$ if $|\{a_\beta\}_{\beta<a}| = 1$;

(ii) $f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<a} = a_0$ if $|\{a_\beta\}_{\beta<a}| > 1$ and $r(a_0, \ldots, a_\beta, \ldots)_{\beta<a}$ holds;

(iii) $f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<a} = a_{\beta_0}$, where $\beta_0 = \min\{\beta: a_\beta \neq a_0\}$ if $|\{a_\beta\}_{\beta<a}| > 1$ and $r(a_0, \ldots, a_\beta, \ldots)_{\beta<a}$ does not hold.

Put $X_0 = \{a \mid r(a, a, \ldots) \text{ holds}\}$, $X_1 = X - X_0$ and define the operation $g_r: X^2 \to X$ as in the proof of the Lemma for the partition $\{X_0, X_1\}$ of the set X.

Let $\varphi \in \mathrm{Aut}(X; r)$. In case (i) we have

$$\varphi f_r(a, a, \ldots) = \varphi a = f_r(\varphi a, \varphi a, \ldots).$$

In case (ii) we have

$$\varphi f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha} = \varphi a_0 = f_r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$$

because $r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$ holds.

Analogously in case (iii) we have

$$\varphi f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha} = \varphi a_{\beta_0} = f_r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$$

because φ is 1-1 and $r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$ does not hold.

Thus we have proved that $\varphi \in \mathrm{Aut}(X; f_r)$.

Next, since, for every $a \in X$, $r(a, a, \ldots)$ holds iff $r(\varphi a, \varphi a, \ldots)$ holds, then $\varphi(X_i) = X_i$ for $i = 0, 1$ and thus we infer that $\varphi \in \mathrm{Aut}(X; f_r, g_r)$ by the Lemma.

Assume now that $\varphi \in \mathrm{Aut}(X; f_r, g_r)$. Then by the Lemma it follows that $\varphi(X_i) = X_i$ for $i = 0, 1$, which means that, for every $a \in X$, $r(a, a, \ldots)$ holds iff $r(\varphi a, \varphi a, \ldots)$ holds.

Suppose now that $|\{a_\beta\}_{\beta<\alpha}| > 1$. If $r(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha}$ holds, then we have

$$\varphi a_0 = \varphi f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha} = f_r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$$

which means that $r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$ holds. Conversely, if the relation $r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha}$ holds, then

$$f_r(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha} = \varphi^{-1} f_r(\varphi a_0, \ldots, \varphi a_\beta, \ldots)_{\beta<\alpha} = \varphi^{-1}\varphi a_0 = a_0$$

which implies that $r(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha}$ holds. Thus we conclude that $\varphi \in \mathrm{Aut}(X; r)$, q.e.d.

COROLLARY. *For any graph* $\mathfrak{L} = (X; r)$, *where* $r \subseteq X^2$, *there exists an algebra* $\mathfrak{A} = (X; f, g)$ *with two binary operations such that* $\mathrm{Aut}\,\mathfrak{L} = \mathrm{Aut}\,\mathfrak{A}$.

2. It is known ([1]-[3]) that for every n-ary relational system $\mathfrak{R}$ there exists an n-ary algebra $\mathfrak{A}$ such that $\mathrm{Aut}\,\mathfrak{R} = \mathrm{Aut}\,\mathfrak{A}$. The following theorem gives some more information about this:

THEOREM 2. *For any relational system* $\mathfrak{R} = (X; \boldsymbol{R})$ *there exists an algebra* $\mathfrak{A} = (X; \{f_r\}_{r\in\boldsymbol{R}}, g)$, *where the operation* g *is binary and the arity of* f_r *is the same as that of* r, *such that* $\mathrm{Aut}\,\mathfrak{A} = \mathrm{Aut}\,\mathfrak{R}$.

Proof. For any $r \in \boldsymbol{R}$ define the operation f_r as in the proof of Theorem 1. Next let us put

$$X_{0r} = \{a \mid r(a, a, \ldots) \text{ holds}\}, \quad X_{1r} = X - X_{0r} \quad \text{for every } r \in \boldsymbol{R}.$$

Thus we have the partitions $\pi_r = \{X_{0r}, X_{1r}\}$, $r \in \boldsymbol{R}$, and $\pi = \bigcap_{r\in\boldsymbol{R}} \pi_r$.

Let the operations g_r, $r \in \boldsymbol{R}$, and g be defined as in the proof of the Lemma for the partitions π_r and π, respectively. Then we have

$$\begin{aligned}\mathrm{Aut}\mathfrak{R} &= \bigcap_{r\in\boldsymbol{R}} \mathrm{Aut}(X; r) = \bigcap_{r\in\boldsymbol{R}} \mathrm{Aut}(X; f_r, g_r)\\ &= \bigcap_{r\in\boldsymbol{R}} \mathrm{Aut}(X; f_r)\cap\bigcap_{r\in\boldsymbol{R}} \mathrm{Aut}(X; g_r)\\ &= \bigcap_{r\in\boldsymbol{R}} \mathrm{Aut}(X; f_r)\cap\mathrm{Aut}(X; g) = \mathrm{Aut}\mathfrak{A},\end{aligned}$$

q.e.d.

Remark. Sometimes we need not define all operations f_r and g. For example, if we have a graph $\mathfrak{L} := (X; r)$, $r \subseteq X^2$, without loops, we need not define g, because g is the first projection. Analogously, if r is included in the diagonal of X^2, then f_r is the second projection.

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