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DOMAIN OF PARTIAL ATTRACTION FOR INFINITELY DIVISIBLE DISTRIBUTIONS IN A HILBERT SPACE

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The paper contains a generalization to a Hilbert space of a theorem of Khintchine [4] which asserts that every one-dimensional infinitely divisible distribution has a domain of partial attraction.

Let p be a probability distribution defined in a separable real Hilbert space H. A sequence p_n of distributions in H is said to be *weakly convergent to a distribution* p ($p_n \to p$) if for any continuous function f bounded in H we have

$$\lim_{n\to\infty} \int f(x) p_n(dx) = \int f(x) p(dx).$$

A sequence of distributions p_n is called *shift-compact* (*convergent*) if there exists a sequence $\{x_n\}$ of elements of H such that the sequence $p_n * \delta_{x_n}$ is compact (convergent) and a distribution p is called *infinitely divisible* if for any natural n there exists a distribution p_n such that $p = p_n^{n*}$, where $*$ denotes the convolution of distributions, p^{n*} denotes the n-th convolution power of p, and δ_x denotes the distribution concentrated at a point $x \in H$.

A distribution p is said *to belong to the domain of partial attraction of a distribution* μ defined in H, if there exists a subsequence of natural numbers $n_1 < n_2 < \ldots < n_r < \ldots$ and a sequence of positive numbers $a_r \to 0$ such that the sequence of measures $(T_{a_r} p)^{n_r *}$ is shift-convergent to μ, where $(T_c p)(Z) = p\{x \in H: cx \in Z\}$ for Z being a Borel subset of H and c a real number.

If the subsequence $\{n_r\}$ coincides with the sequence of all natural numbers, then the distribution p is said *to belong to the domain of attraction of the distribution*.

It is known [3] that only infinitely divisible distributions may have domain of partial attraction.

THEOREM. *Every infinitely divisible distribution in a separable real Hilbert space has a non-empty domain of partial attraction.*

Proof. In the sequel $\hat{\mu}$ stands for the characteristic functional of the measure μ, i.e. $\hat{\mu}(y) = \int e^{i(x,y)}\mu(dx)$, and ${}^0p = p * {}^-p$ with ${}^-p = T_{-1}p$.

Let μ be an infinitely divisible distribution defined in H. Varadhan ([8], Theorem 5.10) has given the general form of the characteristic functional of such a distribution,

$$(1)\quad \hat{\mu}(y) = \exp\left\{i(x_0, y) - \frac{1}{2}(Dy, y) + \int\left[e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|^2}\right]M(dx)\right\},$$

where $x_0 \in H$, D is an S-operator, i.e. a non-negative self-adjoint operator with finite trace, and M is a σ-finite measure in H, which is finite outside every neighbourhood of zero in H, and for which

$$(2)\qquad \int_{\|x\|\leqslant 1} \|x\|^2 M(dx) < +\infty.$$

Every infinitely divisible distribution defined in H is uniquely determined by three elements: $x_0 \in H$, an S-operator and a measure M. In this connection we shall write $\mu = [x_0, D, M]$.

We observe that every infinitely divisible distribution can be written in the form

$$(3)\qquad \mu = \mu_D * \mu_M,$$

where $\mu_D = [0, D, 0]$, $\mu_M = [x_0, 0, M]$.

Let a distribution q, for which

$$(4)\qquad \int\|x\|^2 q(dx) = +\infty,$$

belong to the domain of attraction of the distribution μ_D, i.e. there exists a sequence of positive numbers $a_n \to 0$ such that distribution $(T_{a_n}q)^{n*}$ are shift-convergent to μ_D. Such a distribution does exist (see [1], Chapter VIII, § 4). Hence it follows (see [5], Corollary 3.2) that

$$\lim_{n\to\infty} n \int_{\|x\|\leqslant\varepsilon} (x, y)^2 T_{a_n}{}^0q(dx) = (Dy, y) \quad \text{for every } \varepsilon > 0 \text{ and } y \in H.$$

If $(Dy, y) \not\equiv 0$, there exists an element $y_0 \in H$ for which

$$\int (x, y_0)^2\, {}^0q(dx) = +\infty$$

(see [5], Corollary 3.4) and, consequently,

$$(5)\qquad \lim_{n\to\infty} (na_n^2)^{-1} = +\infty.$$

We introduce the following notation:

$$Q_k = \left\{x \in H : \frac{1}{k} \leqslant \|x\| < k\right\}, \quad k = 1, 2, \ldots,$$

(6) $$M_k = \text{restriction of the measure } M \text{ to } Q_k \quad (M_k \nearrow M),$$

$$V_k = M_k(H).$$

Let a sequence of natural numbers $n_1 < n_2 < \ldots < n_r < \ldots$ increase so fast that

$$2^{n_r} \geqslant r^2 V_r, \tag{7}$$

$$b_{n_r}^{-1} \sum_{k=1}^{r-1} b_{n_k} \int \|x\|^2 M_k(dx) < \frac{1}{r}, \quad r = 1, 2, \ldots, \tag{8}$$

with $n_1 = 1$ and $b_n = (2^{n^2} \cdot a_{2^{n^2}}^2)^{-1}$.

Hence it follows that there exists a strictly increasing subsequence $\{m_r\}$ of natural numbers which satisfies the following conditions:

$$\lim_{r\to\infty} m_r \sum_{k=r+1}^{\infty} m_k^{-1} V_k = 0, \tag{7'}$$

$$m_r a_{m_r}^2 \sum_{k=1}^{r-1} (m_k a_{m_k}^2)^{-1} \int \|x\|^2 M_k(dx) < \frac{1}{r}, \quad r = 1, 2, \ldots \tag{8'}$$

It suffices to put $m_r = 2^{n_r^2}$.

Now we are going to define a distribution which belongs to the domain of partial attraction of the distribution μ_M.

Put

$$\hat{p}(y) = \exp\left\{\sum_{k=1}^{\infty} m_k^{-1} \int [e^{ia_{m_k}^{-1}(x,y)} - 1] M_k(dx)\right\}, \quad y \in H. \tag{9}$$

This formula describes a characteristic functional of a certain probability measure in H. In fact, the functional (9) is obviously positive-definite. Its continuity in the S-topology of Sazonov [7] follows from (7').

We shall prove that the sequence of distributions $(T_{a_{m_r}} p)^{m_r *}$ is shift-convergent to μ_M.

By a similar arguments as in the proof the classical theorem of Khintchine ([2], § 36), we obtain the equality

$$\ln[(T_{a_{m_r}} p)^{m_r *}]\hat{\ }(y) + i(x_r, y) \tag{10}$$

$$= i(x_0, y) + \int \left[e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|^2}\right] M_r(dx) + A_r(y) + B_r(y),$$

where

$$(x_r, y) = (x_0, y) + \int \frac{(x, y)}{1+\|x\|^2} M_r(dx) - m_r \cdot a_{m_r} \sum_{k=1}^{r-1} (m_k a_{m_k})^{-1} \int (x, y) M_k(dx),$$

$$(11)\quad A_r(y) = -\frac{1}{2} m_r a_{m_r}^2 \sum_{k=1}^{r-1} (m_k a_{m_k}^2)^{-1} \int (x, y)^2 e^{i\theta a_{m_r} \cdot a_{m_k}^{-1}(x,y)} M_k(dx), \quad |\theta| < 1,$$

$$B_r(y) = m_r \sum_{k=r+1}^{\infty} m_k^{-1} \int [e^{i a_{m_r} \cdot a_{m_k}^{-1}(x,y)} - 1] M_k(dx).$$

From properties (7′) and (8′) of the sequence $\{m_r\}$ it follows that

$$(12)\qquad \lim_{r\to\infty} A_r(y) = \lim_{r\to\infty} B_r(y) = 0,$$

the convergence being uniform for y belonging to an arbitrary sphere.

Next

$$(13)\quad \lim_{r\to\infty} \int \left[e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|^2}\right] M_r(dx) = \int \left[e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|^2}\right] M(dx),$$

the convergence being uniform for y belonging to an arbitrary sphere (see [8], Theorem 5.7). These facts prove that

$$(14)\qquad \lim_{r\to\infty} [(T_{a_{m_r}} p)^{m_r *} * \delta_{x_r}]\widehat{\ }(y) = \hat{\mu}_M(y),$$

the convergence being uniform for y belonging to an arbitrary sphere.

According to Theorem 5.5 and Corollary 2.3 in [8] it suffices to show that the measures $(T_{a_{m_r}} {}^0p)^{m_r *}$ are compact. We shall prove this by verification of the conditions of Prokhorov theorem ([6], § 4). Estimate:

$$|1 - [(T_{a_{m_r}} {}^0p)^{m_r *}]\widehat{\ }(y)| = 1 - |(T_{a_{m_r}} p)\widehat{\ }(y)|^{2m_r}$$

$$= 1 - \exp\left\{-2m_r \sum_{k=1}^{\infty} m_k^{-1} \int [1 - \cos a_{m_r} a_{m_k}^{-1}(x, y)]\, M_k(dx)\right\}$$

$$\leqslant 2m_r \sum_{k=1}^{\infty} m_k^{-1} \int [1 - \cos a_{m_r} a_{m_k}^{-1}(x, y)] M_k(dx)$$

$$\leqslant m_r a_{m_r}^2 \sum_{k=1}^{r-1} (m_k a_{m_k}^2)^{-1} \int (x, y)^2 M_k(dx) + 2\int [1 - \cos(x, y)] M_r(dx) +$$

$$+ 4m_r \sum_{k=r+1}^{\infty} m_k^{-1} V_k.$$

Since the measures M_r increase to the measure M, we have

$$\int [1-\cos(x, y)]M_r(dx) \leqslant \int [1-\cos(x, y)]M(dx)$$
$$\leqslant \frac{1}{2} \int_{\|x\|\leqslant 1} (x, y)^2 M(dx) + \int_{\|x\|>1} [1-\cos(x, y)]M(dx).$$

The measure M on the set $E = \{x \in H : \|x\| > 1\}$ is tight; thus for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E$ such that $M(K_\varepsilon) \geqslant M(E) - \varepsilon/4$.

We have

$$\int_{\|x\|>1} [1-\cos(x, y)]M(dx) \leqslant \frac{1}{2} \int_{K_\varepsilon} (x, y)^2 M(dx) + \frac{\varepsilon}{2}.$$

Eventually for an arbitrary $\varepsilon > 0$ we have

$$|1-[(T_{a_{m_r}}{}^0 p)^{m_r *}]\hat{}(y)| \leqslant (S_r^\varepsilon y, y) + C_r + \varepsilon, \tag{15}$$

where

$$(S_r^\varepsilon y, y) = m_r a_{m_r}^2 \sum_{k=1}^{r-1} (m_k a_{m_k}^2)^{-1} \int (x, y)^2 M_k(dx) + \tag{16}$$
$$+ \int_{\|x\|\leqslant 1} (x, y)^2 M(dx) + \int_{K_\varepsilon} (x, y)^2 M(dx)$$

and

$$C_r = 4m_r \sum_{k=r+1}^{\infty} m_k^{-1} V_k. \tag{17}$$

Since $\lim_{r\to\infty} C_r = 0$, for an arbitrary $\varepsilon > 0$ and r sufficiently large we have

$$|1-[(T_{a_{m_r}}{}^0 p)^{m_r *}]\hat{}(y)| \leqslant (S_r^\varepsilon y, y) + 2\varepsilon. \tag{18}$$

Let $\{e_i\}$ be an arbitrary orthogonal system defined in H. By standard calculations, (8) and (16) yield the relationships:

$$\sup_r \sum_{i=1}^{\infty} (S_r^\varepsilon e_i, e_i) < +\infty, \quad \lim_{N\to\infty} \sup_r \sum_{i=N}^{\infty} (S_r^\varepsilon e_i, e_i) = 0. \tag{19}$$

Thus the family of S-operators defined by quadratic form (16) is compact. The compactness of the measures $(T_{a_{m_r}}{}^0 p)^{m_r *}$ follows from (18) and (19) by a theorem of Prokhorov [6].

We have shown that the measures $(T_{a_{m_r}} p)^{m_r *}$ are shift-convergent to μ_M. Obviously, the measures $(T_{a_{m_r}} q)^{m_r *}$ are shift-convergent to μ_D. Thus the measures $[T_{a_{m_r}}(p * q)]^{m_r *}$ are shift-convergent to $\mu = \mu_D * \mu_M$, so the proof is complete.

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