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ON INVARIANT FUNCTIONS FOR POSITIVE OPERATORS*

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1. Introduction. Let $(X, \mathscr{A}, \mu)$ be a probability space and let T be a positive linear operator on $L_p(X, \mathscr{A}, \mu)$ for some fixed p, $1 \leqslant p < \infty$. For $p = 1$, the ratio ergodic theorem of Chacon-Ornstein [2] assumes that T is *sub-Markovian*; i.e., $|T|_1 \leqslant 1$. Assuming a weaker boundedness condition (b_h) on T, Sucheston [15] has shown that the space X decomposes into sets Y^h and Z^h such that the ratio ergodic theorem holds on Y^h; Z^h is the largest set which "disappears" under T; moreover, on Y^h there is a positive bounded function e^h which is invariant under T^*, the adjoint of T. We remark that this result can be extended to an operator T on $L_p(X, \mathscr{A}, \mu)$, $1 < p < \infty$ (Theorem 1).

For a sub-Markovian operator T on L_1, Dean and Sucheston [3], and independently Neveu [14], extending results of Mrs. Dowker [5] and Ito [10], have given necessary and sufficient conditions for the existence of positive T-invariant functions in L_1; Krengel [11] has shown that the space X decomposes into a *positive* part P and a *null* part N such that P is the largest set which supports a T-invariant function. We obtain similar results (Proposition 1, Theorems 2 and 3) for an operator T satisfying condition

$$(\mathrm{b}_1)\colon \sup_n |T^n|_1 < \infty .$$

Finally, it has been shown that the ratio ergodic theorem in general does not hold on the disappearing part Z^1 (cf. [9] and [16]). We give an example to show that Hopf's decomposition, which is a consequence of the ratio ergodic theorem, also ceases to be valid on Z^1.

Most results are obtained by the method of Banach limits. It is recalled that Banach limits, or Banach-Mazur limits, are positive linear functionals on the space of bounded sequences of real numbers (x_n),

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satisfying the following axioms:

(i) $L(1) = 1$;

(ii) $L(x_n) \geqslant 0$ if $x_n \geqslant 0$, $n = 0, 1, 2, \ldots$;

(iii) $L(x_n) = L(x_{n+1})$ (shift-invariance).

The maximal value of Banach limits on a sequence (x_n) is (see, e.g., [18])

$$M(x_n) \stackrel{\text{def}}{=} \lim_n \Big(\sup_j n^{-1} \sum_{i=0}^{n-1} x_{i+j}\Big), \tag{1.1}$$

hence the minimal value is $-M(-x_n)$, equal to

$$m(x_n) \stackrel{\text{def}}{=} \lim_n \Big(\inf_j n^{-1} \sum_{i=0}^{n-1} x_{i+j}\Big). \tag{1.2}$$

2. A decomposition of X for L_p-operators. All sets and functions introduced below are assumed measurable; all relations are assumed to hold modulo sets of μ-measure zero. Integrals, unless otherwise indicated, are with respect to the measure μ. The dual space of L_p, $1 \leqslant p < \infty$, is L_q: we assume $1/p + 1/q = 1$. The L_p-norm of f is $|f|_p$; the L_p-norm of an operator T is denoted by $|T|_p$. L_p^+ is the class of non-negative, non-vanishing functions in L_p. We write supp f for the set of points at which the function f is different from zero. $L_p(A)$ is the class of functions f in L_p with supp $f \subset A$. The *potential* operator associated with a positive operator T is denoted by T_∞: for each non-negative function g, $T_\infty g$ is the function $g + Tg + T^2 g + \ldots$ The indicator function of a set A is written 1_A; the function $f \cdot 1_A$ is sometimes written f_A. The adjoint of an operator T is denoted by T^*.

A set A is said to be *closed* (under T) if $f \in L_p(A)$ implies $Tf \in L_p(A)$. A positive operator on L_1 is said to be *conservative* if for each non-negative function g, $T_\infty g = \infty$ or 0.

THEOREM 1. *Let T be a positive linear operator on $L_p(X, \mathscr{A}, \mu)$ such that $\sup_n |T^n|_p < \infty$. Then $X = Y + Z$ and*

(i) *Z is closed;*

(ii) *there is a function $\bar{h} \in L_q^+$ such that if $f \in L_p^+(Y)$, then $M\big[\int T^n f \cdot \bar{h}\big] > 0$;*

(iii) *if $f \in L_p(Z)$, then $M\big[\int T^n |f| \cdot h\big] = 0$ for every $h \in L_q^+$.*

If $X \neq Z$, then there is a function $e \in L_q^+$ with supp $e = Y$ *and $T^* e = e$. If $f \cdot e \in L_1$, $g \cdot e \in L_1^+$, then*

$$D_n(T, f, g) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} T^i f \Big/ \sum_{i=0}^{n-1} T^i g$$

converges to a finite limit on the set $Y \cap$ supp $T_\infty g$. *If there is a function $g \in L_p^+(Z)$ such that the set $C(g) \stackrel{\text{def}}{=} \{T_\infty g = \infty\}$ is non-null, then the ratio theorem fails on every non-null subset of $C(g)$.*

Proof. For the case $p = 1$, when the sets Y, Z are denoted by Y^1, Z^1, the first part of Theorem 1 is due to Sucheston [15], and the last assertion is due to Ionescu Tulcea and Moritz [9]. Since the proof in [15] extends to the case $1 < p < \infty$, we prove here only the last assertion of the theorem. Our argument is mainly that of [9]. Assume that on a non-null subset A of $C(g)$, $D_n(T, f, g)$ converges to a finite limit a.e. for every $f \in L_p$. Let S be the operator from L_p into $\mathscr{M}$, the space of real-valued measurable functions on $(X, \mathscr{A})$, defined by

$$Sf(x) = 1_A(x) \cdot \lim_n D_n(T, f, g)(x).$$

Let

$$g_n = n^{-1} \sum_{i=0}^{n-1} T^i g, \quad n = 1, 2, \ldots$$

Since $T_\infty g = \infty$ on A, we have

$$Sg_n(x) = \lim_m \left[n^{-1} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} T^{i+j} g(x) \right] \Big/ \left[\sum_{i=0}^{m-1} T^i g(x) \right] \geqslant 1 \tag{2.1}$$

on A. It follows from (iii) that $\lim_n \int g_n \cdot h = 0$ for every $h \in L_q^+$; i.e., g_n converges weakly to zero. A mean ergodic theorem for Banach spaces ([6], p. 661) implies that $\lim_n |g_n|_p = 0$. Thus we can choose a subsequence (g_{n_i}) with $f = \sum_{i=1}^{\infty} g_{n_i} \in L_p$. Then

$$0 \leqslant \sum_{i=1}^{\infty} Sg_{n_i} \leqslant Sf < \infty \quad \mu\text{-a.e.};$$

hence $\lim_i Sg_{n_i} = 0$ μ-a.e., but this contradicts (2.1).

3. T-Invariant functions in L_1. For $1 < p < \infty$, L_p is a reflexive Banach space, and hence the problem of existence of T-invariant functions in L_p is reduced to one of existence of T^*-invariant functions in L_q. Henceforth, we shall assume that T is a positive linear operator on L_1 satisfying condition (b_1), which is equivalent with

$$\sup_n |T^n|_1 < \infty. \tag{b_1}$$

Such an operator has been called *semi-Markovian* [15]. T admits an adjoint operator T^* which acts on L_∞; the adjoint T^{**} of T^* operates on the space Ψ of signed finite finitely additive set functions vanishing on μ-null sets (cf. [6], p. 296). Under the natural embedding of the Banach space L_1 in its second conjugate Ψ, L_1 is mapped on Φ, the space of finite signed μ-continuous measures. If $\nu \in \Phi$, then $T^{**}\nu \in \Phi$ and

$$(3.1) \qquad T^{**}\nu(A) = \int_A T\frac{d\nu}{d\mu}\,d\mu, \qquad A \in \mathscr{A}.$$

We shall often write $T\nu$ for $T^{**}\nu$ if $\nu \in \Phi$. If $\nu = \mu$, then $d\nu/d\mu = 1$, and we have for each $n \geqslant 0$

$$(3.2) \qquad T^n\mu(A) = \int_A T^n 1\,d\mu = \int T^{*n}1_A\,d\mu, \qquad A \in \mathscr{A}.$$

PROPOSITION 1. *Let T be a semi-Markovian operator on L_1. Then X is the disjoint union of two uniquely determined sets P and N with the following properties*:

(a) $A \subset P$, $\mu(A) > 0$ *implies* $M[T^n\mu(A)] > 0$;

(b) *N is the disjoint union of countably many sets X_i with* $M[T^n\mu(X_i)] = 0$ *for each i*;

(c) *P is closed under T.*

In the sub-Markovian case, Proposition 1 has been proved by Krengel [11]; Dean and Sucheston previously showed that $X = N$ in the particular case where T is assumed conservative, ergodic, and has no positive fixed-point ([3], Theorem 2).

Proof. Let $H(A) = M[T^n\mu(A)]$, $A \in \mathscr{A}$.

It is clear that $H(\emptyset) = 0$ and that if $A \subset B$, then $H(A) \leqslant H(B)$. Let X_i, $i = 1, 2, \ldots$, be a disjoint sequence of sets such that $H(X_i) = 0$ for each i and

$$\lim_n \mu\Big(\bigcup_{i=1}^n X_i\Big) = \sup_{H(A)=0} \mu(A).$$

Set $N = \bigcup_{i=1}^\infty X_i$ and $P = X - N$. It is then easy to verify that P and N satisfy (a) and (b). We now prove (c); our argument is simpler than Krengel's in [11]. Since P is T-closed if and only if its complement N is T^*-closed, we need only to show that $T^*1_N = 0$ on P. By the monotone continuity property of T^* (cf. Neveu [13], p. 187), we have

$$T^*1_N = \lim_n T^*\Big(\sum_{i=1}^n 1_{X_i}\Big);$$

thus, it is sufficient to show that $T^*1_{X_i} = 0$ on P for each i. Assume $T^*1_{X_i} \neq 0$ on P for some i. Then there is an $\varepsilon > 0$ and a set $A \subset P$ with $\mu(A) > 0$ such that $T^*1_{X_i} \geqslant \varepsilon$ on A. It follows that $T^{*n+1}1_{X_i} \geqslant \varepsilon \cdot T^{*n}1_A$ for every n. This yields the contradiction $M[T^n\mu(X_i)] \geqslant \varepsilon \cdot M[T^n\mu(A)] > 0$. Thus, P is T-closed.

We now assume $X = Y^1$; for the definition of Y^1 see Section 2. Thus, there is a bounded function e, $0 < e \leqslant 1$, such that $T^* e = e$; in the sequel, e will be assumed chosen and fixed. We introduce an auxiliary operator V on L_1 by the relation

$$(3.3) \qquad Vf = e\cdot T(f/e). \quad f \in L_1^+,$$

and define Vf by linearity for $f \in L_1$. V is then a positive linear contraction on L_1, and for each n, $V^n f = e\cdot T^n(f/e)$ (see [15], p. 4). Proposition 1 applied to T and V gives the decompositions $X = P_T + N_T$ and $X = P_V + N_V$.

LEMMA 1. *Assume $X = Y^1$ and let V be defined by* (3.3). *Then the decompositions $X = P_T + N_T$ and $X = P_V + N_V$ coincide.*

Proof. For each $\varepsilon > 0$, let $E_\varepsilon = \{e < \varepsilon\}$. Let $A \subset P_V$ with $\mu(A) > 0$. Since $|V|_1 \leqslant 1$ and $0 < e \leqslant 1$, we have

$$\begin{aligned} V^n\mu(A) &= \int_A V^n 1 = \int_A V^n 1_{E_\varepsilon} + \int_A V^n 1_{E_\varepsilon^c} \\ &\leqslant \int_X V^n 1_{E_\varepsilon} + \int_A e\cdot T^n(1_{E_\varepsilon^c}/e) \\ &\leqslant \mu(E_\varepsilon) + (1/\varepsilon)\cdot \int_A T^n 1_{E_\varepsilon^c} \leqslant \mu(E_\varepsilon) + (1/\varepsilon)\cdot \int_A T^n 1 \end{aligned}$$

for each $\varepsilon > 0$ and each n. Since $e > 0$, for each $\delta > 0$ there is an $\varepsilon > 0$ such that $\mu(E_\varepsilon) < \delta$, and hence

$$(3.4) \qquad V^n\mu(A) < \delta + (1/\varepsilon)\cdot T^n\mu(A)$$

which implies $M[T^n\mu(A)] > 0$. It easily follows that $A \subset P_T$ and $P_V \subset P_T$.

We next let $A \subset P_T$ with $\mu(A) > 0$. Since $\mu(E_\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists an $\varepsilon > 0$ such that $\mu(A \cap E_\varepsilon^c) > 0$, and

$$\begin{aligned} V^n\mu(A) \geqslant V^n\mu(A \cap E_\varepsilon^c) &= \int_{A\cap E_\varepsilon^c} e\cdot T^n(1/e) \\ &\geqslant \varepsilon\cdot \int_{A\cap E_\varepsilon^c} T^n 1 = \varepsilon\cdot T^n\mu(A \cap E_\varepsilon^c) \end{aligned}$$

for each n. It follows that $M[V^n\mu(A)] \geqslant \varepsilon\cdot M[T^n\mu(A \cap E_\varepsilon^c)] > 0$ and $A \subset P_V$. Hence, $P_T \subset P_V$.

For a sub-Markovian operator T on L_1, the following conditions (o), (i), (ii) have been proved to be equivalent in [3] and [14]. Here we replace $|T|_1 \leqslant 1$ by (b_1).

THEOREM 2. *Let T be a semi-Markovian operator on L_1 and assume $X = Y^1$. Then the following conditions are equivalent*:

(o) *There exists a function $f \in L_1$ with $f > 0$ and $Tf = f$.*

(i) $\mu(A) > 0$ *implies* $\inf_n T^n\mu(A) > 0$, $A \in \mathscr{A}$.

(ii) $\mu(A) > 0$ *implies* $M[T^n\mu(A)] > 0$, $A \in \mathscr{A}$.

Proof. We introduce a third condition:

(iii) *$\mu(A) > 0$ implies $m[T^n\mu(A)] > 0$, $A \in \mathscr{A}$.*

Our proof follows the scheme (o) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (o). The implication (i) $\Rightarrow$ (iii) is obvious.

Part I. (o) $\Rightarrow$ (ii). Assume that there is a function $f \in L_1$ such that $f > 0$ and $Tf = f$. Let $g = f\cdot e$; then $g \in L_1, g > 0$, and $Vg = g$. Since V is sub-Markovian, by [3] or [14], $X = P_V$; thus, by Lemma 1, $X = P_T$.

Part II. (ii) $\Rightarrow$ (i). Assume condition (ii); i.e., $X = P_T$. By Lemma 1, $X = P_V$; since $|V|_1 \leqslant 1$, the condition $X = P_V$ is equivalent ([3] or [14]) to

$$(3.5) \qquad \mu(A) > 0 \quad \text{implies} \quad \inf_n V^n\mu(A) > 0, \qquad A \in \mathscr{A}.$$

Relations (3.4i) and (3.5) together imply that condition (i) holds.

Part III. (iii) $\Rightarrow$ (o). In [3], the authors showed that (iii) implies that there exists a measure ν equivalent with μ with $T^{**}\nu \leqslant \nu$; their proof does not assume $|T|_1 \leqslant 1$. Let $f = d\nu/d\mu$; then $f \in L_1, f > 0$, and $Tf \leqslant f$. The following standard argument shows that if T is conservative on L_1, then $Tf = f$. Indeed, let $g = f - Tf$; we have $g \in L_1, g \geqslant 0$, and

$$\int \sum_{i=0}^{n-1} T^i g = \int f - \int T^n f \leqslant \int f < \infty$$

for every n. Thus, $g = 0$. It remains only to show that T is conservative. Since Hopf's decomposition is valid on Y^1, hence on X (cf. [15], Theorem 2), it is sufficient to show that $T_\infty 1 = \infty$ on X. Since (iii) $\Rightarrow$ (ii), for each set E with $\mu(E) > 0$, we have

$$\int M[T^{*n}1_E] \geqslant M\Big[\int T^{*n}1_E\Big] = M\Big[\int_E T^n 1\Big] > 0.$$

It follows that $\sum_{n=0}^{\infty} T^{*n}1_E = \infty$ on a set of positive μ-measure. If $\{T_\infty 1 < \infty\} \neq \emptyset$, then there is a constant $a \geqslant 0$ such that $\mu\{T_\infty 1 \leqslant a\} > 0$. Set $E = \{T_\infty 1 \leqslant a\}$; then

$$(3.6) \quad \int \sum_{i=0}^{n} T^{*i}1_E = \int_E \sum_{i=0}^{n} T^i 1 \leqslant a\cdot\mu(E) < \infty, \qquad n = 0, 1, 2, \ldots,$$

which is a contradiction since the first term in (3.6) tends to infinity as $n \to \infty$.

PROPOSITION 2. *Let T be a semi-Markovian operator on L_1, satisfying condition* (b$_1$). *Assume $X = Y^1$. Then there is a non-negative function $f \in L_1$ with $f > 0$ on P and $Tf = f$.*

Proof. Assume $X = Y^1$; by Lemma 1, $P_T = P_V$, which we denote simply by P. Define an operator T' on $L_1(P, P \cap \mathscr{A}, \mu)$ by the relation

$$(3.7) \qquad T'f = Tf, \quad f \in L_1(P).$$

Clearly, T' satisfies condition (b_1) on $L_1(P)$. The adjoint T'^* of T' is given by the relation

$$(3.8) \qquad T'^* h = 1_P \cdot T^* h, \quad h \in L_\infty(P).$$

V' and V'^* are defined similarly. Proposition 1 applied to T' and V' gives the decompositions $P = P_{T'} + N_{T'}$ and $P = P_{V'} + N_{V'}$. Since $|V|_1 \leqslant 1$ on L_1, there is a non-negative function $f_0 \in L_1$ with $f_0 > 0$ on P and $Vf_0 = f_0$ (cf. Krengel [11], Theorem 1); equivalently, f_0 is a positive invariant function in $L_1(P)$ for the sub-Markovian operator V'. Hence $P = P_{V'}$. Recalling that $T^* e = T^*(e_P + e_N) = e_P + e_N$ and that N is T^*-closed, we have $T'^* e_P = 1_P \cdot T^* e_P = e_P$. Thus the space P is seen to be Y^1 for T'. Moreover, since

$$V'f = Vf = e \cdot T(f/e) = e_P \cdot T'(f/e_P), \quad f \in L_1^+(P),$$

Lemma 1 shows that T' and V' give rise to identical decompositions: $P = P_{V'} = P_{T'}$. Theorem 2 applied to T' shows that there is a function $f \in L_1(P)$ with $f > 0$ and $T'f = f$. This proves the proposition.

The following theorem, due in the sub-Markovian case to Dean and Sucheston [3], relates the existence of T-invariant functions to the uniqueness of Banach limits on sequences $T^n \mu(A)$.

THEOREM 3. *Let T be a semi-Markovian operator on L_1. Assume $X = Y^1$. If T has a positive invariant function, then for each set A all Banach limits of the sequence $T^m \mu(A)$ coincide; if $\lambda(A)$ is their common value, then*

$$(3.9) \qquad \lambda(A) = \lim_n n^{-1} \sum_{i=0}^{n-1} T^{i+j} \mu(A) \quad \textit{uniformly in } j,$$

and $d\lambda/d\mu$ is a positive invariant function. Conversely, if for each set A, all Banach limits on the sequence $T^n \mu(A)$ coincide and if T is conservative, then T has a positive invariant function.

Proof. Assume that there is a positive invariant function f_0. Hopf's decomposition holds on Y^1 (cf. [15], Theorem 2), and hence on X; thus T is conservative since $\{T_\infty f_0 = \infty\} = X$. Let L be a Banach limit and set

$$(3.10) \qquad \lambda(h) = L[\textstyle\int T^{*n} h], \quad h \in L_\infty.$$

It is clear that λ defines a positive continuous linear functional on L_∞.

Writing $\lambda(A)$ for $\lambda(1_A)$, $A \in \mathscr{A}$, λ is seen to be a μ-continuous measure (see [3], Theorem 3). Let $f = d\lambda/d\mu$. Then $f \in L_1$,

$$\lambda(h) = \int f\cdot h, \quad h \in L_\infty, \tag{3.11}$$

and it follows from the shift-invariance of L that $Tf = f$. Set $F = \{f = 0\}$; then $\mu(F) = 0$, for otherwise it would follow from Theorem 2 that $\lambda(F) > 0$. Hence f is a positive invariant function. If L' is another Banach limit, then the same argument shows that

$$\lambda'(h) = L'\left[\int T^{*n}h\right] = \int f'\cdot h, \quad h \in L_\infty, \tag{3.12}$$

where $f' \in L_1$, $f' > 0$, and $Tf' = f'$. Since $X = Y^1$ and T is conservative, the ratio ergodic theorem shows that

$$f = \lim n^{-1} \sum_{i=0}^{n-1} T^i f = \frac{E(f\cdot e/\mathscr{C})}{E(f'\cdot e/\mathscr{C})}\cdot f', \tag{3.13}$$

where $\mathscr{C}$ is the σ-algebra of T-closed, and hence T^*-closed, sets (cf. [15], Theorem 2). Thus $f = g\cdot f'$, where g is a $\mathscr{C}$-measurable function. If $A \in \mathscr{C}$, then both A and A^c are T^*-closed; thus $T^*e_A = e_A$ and

$$\int_A g\cdot f'\cdot e = \lambda(e_A) = \lambda'(e_A) = \int_A f'\cdot e, \quad A \in \mathscr{C}. \tag{3.14}$$

Since $e\cdot f' > 0$ and g is $\mathscr{C}$-measurable, (3.14) shows that $g = 1$ and $f = f'$. This proves the first part of the theorem. To prove the second part, we proceed as in [3] to obtain a μ-continuous measure λ, where $\lambda(A)$ is the Cesàro limit of the sequence $T^n\mu(A)$. Let $f = d\lambda/d\mu$; it is easy to see that $Tf = f$. Set $F = \{f = 0\}$; then both F and F^c are T^*-closed, and thus $T^*e_F = e_F$. If $\mu(F) > 0$, then we arrive at the contradiction:

$$\lambda(F) \geqslant \liminf_n n^{-1} \sum_{i=0}^{n-1} \int T^{*i} e_F = \int e_F > 0.$$

Thus, f is a positive invariant function of T.

4. An example. Let $(X, \mathscr{A}, \mu)$ be a discrete measure space, where $X = \{0, 1, 2, \ldots\}$ and μ is the counting measure on $\mathscr{A}$. A function $f = (f_0, f_1, f_2, \ldots)$ is in L_1 if and only if $\sum |f_i| < \infty$. Define an operator T on L_1 by

$$(Tf)_j = \begin{cases} \sum_{i=1}^{\infty} f_i, & j = 0, \\ f_{j+1}, & j \geqslant 1. \end{cases}$$

It follows that for each $n \geqslant 0$,

$$(T^n f)_j = \begin{cases} \sum_{i=n}^{\infty} f_i, & j = 0, \\ f_{j+n}, & j \geqslant 1, \end{cases}$$

and that

$$|T^n f|_1 \leqslant 2 \cdot \sum_{i=n}^{\infty} |f_i| \leqslant 2 \cdot |f|_1.$$

Thus $\sup_n |T^n|_1 \leqslant 2$ and $\lim_n |T^n f|_1 = 0$ for every $f \in L_1$; hence, $X = Z^1$. Consider the function $g = (g_i)$ with $g_i = 1/i^2, i \geqslant 1$, and $g_0 = 0$. Then $g \in L_1^+$, and $(T_\infty g)_0 = \infty$. On the other hand, for every non-negative, non-vanishing function f which vanishes on all but a finite of points we have $0 < (T_\infty f)_0 < \infty$. Hence Hopf's decomposition does not hold on the singleton set $\{0\}$.

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