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MONOTONE RETRACTS OF AN ARCWISE CONNECTED CONTINUUM

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A *continuum* means a non-degenerate compact connected metric space. A continuum M is *irreducible* between its points p and q if no proper subcontinuum of M contains both p and q. A continuum M is *hereditarily unicoherent* at p if the intersection of any two subcontinua containing p is connected.

PROPOSITION 1 ([1], Theorem 1.3). *A continuum M is hereditarily unicoherent at p if and only if, for any given $x \in M$, there exists a unique subcontinuum which is irreducible between p and x.*

A *dendroid* is an arcwise connected continuum which is hereditarily unicoherent at every point. It is known

PROPOSITION 2 ([2], Theorems 2.2 and 2.3; [7], Theorem 2.6). *An arcwise connected continuum M is a dendroid if and only if M is hereditarily unicoherent at some point p.*

A subcontinuum N of a continuum M is called a *monotone retract* of M if there exists a mapping r from M onto N which is both monotone and a retraction.

THEOREM 1. *Let M be an arcwise connected continuum. If each subcontinuum of M which is irreducible between a fixed point p and some other point is a monotone retract of M, then M is a dendroid.*

Proof. It suffices to show, by Proposition 2, that M is hereditarily unicoherent at p. Observe first that

(1) If $I(p, x)$ is a continuum in X which is irreducible between p and x, then $I(p, x)$ is an arc.

In fact, there is a retraction r from M onto $I(p, x)$. Since M is arcwise connected, so is $I(p, x)$. Therefore, there is an arc px in $I(p, x)$ joining points p and x. But then $I(p, x) = px$ by the irreducibility of $I(p, x)$ between p and x.

(2) Every subcontinuum of M is arcwise connected.

Indeed, let pq be an arc in M joining points p and q and such that $pq \cap Q = \{q\}$, where Q is an arbitrary subcontinuum of M. Take points x and y in Q and consider continua $I(q, x)$ and $I(q, y)$ in Q irreducible between q and x, and q and y, respectively. Sets $pq \cup I(q, x)$ and $pq \cup I(q, y)$ are irreducible between p and x, and p and y, respectively. By (1), they are arcs. Therefore, also continua $I(q, x)$ and $I(q, y)$ are arcs. Thus the continuum $I(q, x) \cup I(q, y)$ contains an arc xy joining x and y. But $I(q, x) \cup I(q, y) \subset Q$, whence $xy \subset Q$. This means that Q is an arcwise connected continuum.

Now suppose, on the contrary, that M is not hereditarily unicoherent at p. Then there are, by Proposition 1 and (1), a point z and two different arcs A and B both joining points p and z. The union $A \cup B$ contains a simple closed curve S. Let pa be an arc in M joining points p and a and such that $pa \cap S = \{a\}$. Take a point b in $S \setminus \{a\}$ and denote two possible arcs in S, both joining points a and b, by ab and $I(a, b)$.

The set $pa \cup ab$ is a continuum irreducible between p and b. Thus there is a monotone retraction r from M onto $pa \cup ab$. We will show that

(3) The arc $I(a, b)$ contains an arc cd such that $r(cd)$ is a non-degenerate arc $r(c)r(d)$ contained in the set $ab \setminus \{a, b\}$ and

$$r^{-1}r(\{c, d\}) \cap cd = \{c, d\}.$$

Since $ab \subset r(I(a, b))$, we infer, using Brouwer's reduction theorem, that there is an arc $c'd'$ in $I(a, b)$ which is irreducible with respect to the property that $ab \subset r(c'd')$. Then

$$r^{-1}(\{a, b\}) \cap c'd' = \{c', d'\}.$$

Now, let a' and b' be two different points belonging to the set $ab \setminus \{a, b\}$. Since $ab \subset r(c'd')$, we conclude that

$$r^{-1}(a') \cap c'd' \neq \emptyset \quad \text{and} \quad r^{-1}(b') \cap c'd' \neq \emptyset.$$

Taking an arc cd irreducible between sets $r^{-1}(a') \cap c'd'$ and $r^{-1}(b') \cap c'd'$ and contained in $c'd'$ we see that cd satisfies the required conditions.

We may assume that $p \leqslant a < r(c) < r(d) < b$ in the natural order of the arc $pa \cup ab$ from a to b, and let $\{x_n\}$ be a sequence of points such that

(4) $\lim x_n = r(d)$

and, in the same order,

(5) $r(c) < x_1 < x_2 < \ldots < r(d)$.

Since $r(c)r(d) \subset r(cd)$, we infer that $r^{-1}(x_n) \cap cd \neq \emptyset$ for each $n = 1, 2, \ldots$ Sets $r^{-1}(x_n)$ are continua and

$$r^{-1}(x_n) \cap (pa \cup ab) = \{x_n\} \quad \text{for each } n = 1, 2, \ldots,$$

since r is a monotone retraction from M onto $pa \cup ab$. Thus for each $n = 1, 2, \ldots$ there is an arc $x_n y_n$ such that

(6) $x_n y_n \cap cd = \{y_n\}$ and $x_n y_n \subset r^{-1}(x_n)$.

Since r is continuous, we obtain

(7) $\lim y_n = d$.

Moreover, considering suitable subsequences of the sequence $\{x_n\}$ we may assume that

(8) the sequence $\{x_n y_n\}$ is convergent

and, in the natural order of cd from c to d,

(9) $c < y_1 < y_2 < \ldots < d$.

Denote the continuum $\mathrm{Lim}\, x_n y_n$ by K. Then, by (4), (6) and continuity of r,

(10) $K \subset r^{-1} r(d)$.

Now let $x_i x_j$ be an arc in ab, let $y_i y_j$ be an arc in cd for $i, j = 1, 2, \ldots,$ and let ax_1 be an arc in ab. Put $C_0 = ax_1$ and $C_n = x_n x_{n+1}$ if n is even and $C_n = y_n y_{n+1}$ if n is odd. From (5) and (9) we obtain

(11) $C_n \cap C_m = \emptyset$ for $n \neq m$ and $n, m = 0, 1, 2, \ldots$

and, by (4) and (7),

(12) $\mathrm{Lim}\, C_{2n} = r(d)$, $\mathrm{Lim}\, C_{2n+1} = d$.

Consider the set

$$L = pa \cup \bigcup_{n=0}^{\infty} (C_n \cup x_{n+1} y_{n+1}).$$

Since

$$pa \cap C_0 = \{a\} \quad \text{and} \quad x_{n+1} y_{n+1} \cap C_n \neq \emptyset \neq x_{n+1} y_{n+1} \cap C_{n+1},$$

we conclude that

(13) the set L is connected.

We have

(14) $\bar{L} \setminus L = K$.

Indeed, if $z \in \bar{L} \setminus L$, then

$$z \in \mathrm{Lim}\, x_n y_n \cup \mathrm{Lim}\, C_{2n} \cup \mathrm{Lim}\, C_{2n+1} = K \cup \{r(d)\} \cup \{d\} = K$$

by (4), (7) and (12). Thus $\bar{L} \setminus L \subset K$. But $K \subset \bar{L}$ since

$$K = \mathrm{Lim}\, x_n y_n \subset \bar{L}.$$

Thus to prove (14) it suffices to show that $K \cap L = \emptyset$. Since $K \subset r^{-1}r(d)$ (cf. (10)) and $r^{-1}r(d) \cap cd = \{d\}$ (cf. (3)), we have $K \cap cd \subset \{d\}$. And since r is a retraction onto $pa \cup ab$, we infer, by (10), that

$$K \cap (pa \cup ab) \subset \{r(d)\}.$$

Thus $(pa \cup C_n) \cap K = \emptyset$ for each $n = 0, 1, 2, \ldots$, since every C_n is contained in $(pa \cup ab \cup cd) \setminus \{r(d), d\}$ (cf. (5) and (9)). Moreover, since $x_n y_n \subset r^{-1}(x_n)$ and $K \subset r^{-1}r(d)$ (see (6) and (10)) and $x_n \neq r(d)$ by (5), we have $x_n y_n \cap K = \emptyset$ for each $n = 1, 2, \ldots$ Therefore $L \cap K = \emptyset$.

From (13) and (14) we infer that

(15) the set $L \cup K$ is a continuum containing p.

Now,

(16) if

$$e \in \bigcup_{n=1}^{\infty} x_n y_n,$$

then the set $(L \cup K) \setminus \{e\}$ is the union of two connected sets U and V such that $p \in U$, $K \subset V$ and $\overline{U} \cap \overline{V} = \{e\}$.

Indeed, let $e \in x_n y_n$ and let ex_n and ey_n be arcs in $x_n y_n$. Put

$$U' = pa \cup C_0 \cup \bigcup_{i=1}^{n-1} (C_i \cup x_i y_i) \quad \text{and} \quad V' = K \cup \bigcup_{i=n}^{\infty} (C_i \cup x_{i+1} y_{i+1}).$$

If n is even, then we put

$$U = U' \cup (ey_n \setminus \{e\}) \quad \text{and} \quad V = V' \cup (ex_n \setminus \{e\}),$$

and if n is odd, then we set

$$U = U' \cup (ex_n \setminus \{e\}) \quad \text{and} \quad V = V' \cup (ey_n \setminus \{e\}).$$

The same arguments as in the proof of (13) show that sets defined in this way are connected. Since $x_n y_n \subset r^{-1}(x_n)$ and $x_n \neq y_n$, we infer that $U \cap V = \emptyset$ by (11) and (14). But $\overline{U} = U \cup \{e\}$ and $\overline{V} = V \cup \{e\}$, thus $\overline{U} \cap \overline{V} = \{e\}$.

Finally, we infer from (2) and (15) that there is an arc pk in $L \cup K$ such that $pk \cap K = \{k\}$. But, by (16), $\{x_n, y_n\} \subset pk$ for each $n = 1, 2, \ldots$ Therefore, by (4) and (7), $\{d, r(d)\} \subset pk$, a contradiction, since

$$\{d, r(d)\} \subset \operatorname{Lim} x_n y_n = K \quad \text{and} \quad d \neq r(d).$$

Thus the proof of Theorem 1 is completed.

Theorem 1 in this paper and Theorem 3 in [6] yield the following

COROLLARY 1. *Let M be an arcwise connected continuum and let $p \in M$. Then M is a dendrite if and only if each subcontinuum of M which is irreducible between p and some other point is a monotone retract of M.*

Thus, by Theorem in [3], we also have

COROLLARY 2. *Let a continuum M be arcwise connected and let $p \in M$. Then the following conditions are equivalent:*

(i) *each subcontinuum of M containing p is a monotone retract of M;*

(ii) *each subcontinuum of M which is irreducible between p and some other point is a monotone retract of M;*

(iii) *each subcontinuum of M is a monotone retract of M.*

Corollary 2 gives a partial solution to a problem asked in [4]: Is it true that (i) and (ii) of Corollary 2 are equivalent for any continuum M?

THEOREM 2. *Let a continuum M be arcwise connected and let $p \in M$. Then M is a dendrite if and only if each subcontinuum of M with a non-empty interior and containing p is a monotone retract of M.*

Proof. If M is a dendrite, then each subcontinuum of M is a retract of M by Corollaries 1 and 2.

Suppose now that each subcontinuum of M with a non-empty interior and containing p is a monotone retract of M. Observe first that

(17) every subcontinuum of M with a non-empty interior is a monotone retract of M.

In fact, let Q be a subcontinuum of M with a non-empty interior and let pq be an arc in M such that $pq \cap Q = \{q\}$. Then the set $pq \cup Q$ is a subcontinuum of M with a non-empty interior and containing p. Therefore, there is a monotone retraction r from M onto $pq \cup Q$. But

$$f(x) = \begin{cases} x & \text{if } x \in Q, \\ q & \text{if } x \in pq \end{cases}$$

is a monotone retraction from $pq \cup Q$ onto q. Then the composition fr is a monotone retraction from M onto Q.

Secondly, we prove that

(18) M is unicoherent.

Indeed, if M is not unicoherent, then there are subcontinua Q and R such that $M = Q \cup R$ and the set $Q \cap R$ is not connected. Then Q (similarly R) is a subcontinuum of M with a non-empty interior. Therefore, by (17), there is a monotone retraction from M onto Q. Choose $y \in r(R) \setminus R$. Then $r^{-1}(y)$ is a subcontinuum which intersects $Q \setminus R$ and $R \setminus Q$. Consequently, $r^{-1}(y)$ meets $Q \cap R$, contradicting the fact that r is fixed in $Q \cap R$.

Since the unicoherence is an invariant under monotone mappings, every subcontinuum of M with a non-empty interior is unicoherent by (17) and (18). But then the arcwise connectivity of M implies (see [9], Corollary 2) that

(19) M is a dendroid.

Now we prove that

(20) if x and y are different points of M, then M is locally connected either at x or at y.

It follows from Theorem 1 in [8] that there is a subcontinuum Q of X with a non-empty interior such that Q contains one of these points and fails to contain the other. Say, $x \in Q \subset M \setminus \{y\}$. Suppose that M is not locally connected at y. Then there is a closed neighbourhood E of y such that $E \cap Q = \emptyset$, and if C is the component of y in E, then y does not belong to its interior, $y \in \overline{E \setminus C}$. Let

(21) $y = \lim y_n, \ y_n \in E \setminus C$.

Since M is arcwise connected, there is an arc ab such that $ab \cap Q = \{a\}$ and $ab \cap C = \{b\}$. Then

(22) $y \in M \setminus (ab \cup Q)$.

Now, consider the continuum $Q \cup ab \cup C$. This continuum has a non-empty interior in M. Thus, by (17), there exists a monotone retraction r from M onto $Q \cup ab \cup C$. Observe that

(23) $r^{-1}(y)$ is degenerate.

In fact, since $r(z) = z$ for $z \in C$, we conclude that $r^{-1}(y) \cap C = \{y\}$. The set $r^{-1}(y)$ is a continuum. If it were non-degenerate, then (by Theorem 4 in [5], § 47, III, p. 173) there would exist a non-degenerate subcontinuum N of $r^{-1}(y)$ containing y and contained in the interior of E, since y belongs to the interior of E and $y \in r^{-1}(y)$. But $y \in C$ and C is a component of E, thus $N \subset C$. Hence $N \subset r^{-1}(y) \cap C$, a contradiction.

It follows from (21) that, by the continuity of r,

(24) $\operatorname{Lim} r^{-1} r(y_n) = \{y\}$.

Since $r(y_n) \in Q \cup ab \cup C$, $\lim r(y_n) = r(y) = y$ by (21) and $y \in C \setminus (Q \cup ab)$ by (22), we obtain $r(y_n) \in C$ with an arbitrarily large n. Then

$$r(y_n) \in r^{-1} r(y_n) \cap C.$$

Thus, by (24), the set $r^{-1} r(y_n)$ with an arbitrarily large n is contained in C, since $r^{-1} r(y_n)$ is connected, C is a component of y in E, and y belongs to the interior of E. But $y_n \in r^{-1} r(y_n) \subset C$, which is impossible by (21). Therefore, the proof of (20) is completed.

(25) M is locally connected.

Fix $x \in M$. If M is not locally connected at x, then, by (20), M is locally connected at any point $y \in M \setminus \{x\}$. But this is impossible. Thus M is locally connected at x.

By virtue of (19) and (25), M is a dendrite.

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