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SEPARATION CONDITIONS IN RELATIVELY COMPLEMENTED LATTICES

BY

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1. Introduction. Holland, Jr. [7], p. 341, lists a number of statements which are true for the projection lattice of a von Neumann algebra or for a complete complemented modular lattice — none of which, however, is true for an arbitrary orthomodular lattice. Included therein is the statement that the center of each interval $[0, a]$ is the set of all $z \wedge a$ with z central in L. During the course of an investigation of *quantifiers* on an orthomodular lattice ([8], p. 100-105), we came on this same condition independently and from totally different considerations. It therefore seemed reasonable to initiate an investigation into the meaning of the condition. This in turn has led us to the consideration of various "separation" conditions of an orthomodular lattice — indeed, of an arbitrary relatively complemented lattice with 0 and 1.

In section 2 we discuss separation conditions in the setting of a relatively complemented lattice, while in section 3 the results are specialized to the case of an orthomodular lattice. Finally, in section 4 we discuss the relation between the center of an interval sublattice and the center of the entire lattice. Some of the material in sections 3 and 4 has become part of the "folklore" of orthomodular lattice theory. We present it here for two reasons: (α) to show its relation to the separation conditions of section 2; (β) to make it accessible for the first time to the general mathematical public.

2. Separation conditions. *In this section L will always denote a relatively complemented lattice with* 0 *and* 1.

Definition 2.1. Two elements e, f of L are called *perspective*, denoted $e \sim f$, in case they have a common complement in L; for each positive integer i, they are called (i)-*perspective*, in symbols $e \sim^{(i)} f$, in case there exist elements $e_1, \ldots, e_i$ such that $e \sim e_1 \sim \ldots \sim e_i = f$; finally, they are called *projective* and denoted $e \approx f$ if they are (i)-perspective for some positive integer i.

Definition 2.2. (i) Let R be a binary relation on L. Two elements e, f of L are called *unrelated with respect to* R if $e_1 \leqslant e, f_1 \leqslant f$ with $e_1 R f_1$ implies $e_1 = f_1 = 0$.

(ii) For each positive integer i, write $eS^{(i)}f$ in case e, f are unrelated with respect to (i)-perspectivity, and $eS^{(\infty)}f$ to denote their being unrelated with respect to projectivity. Conditions of this type will be referred to in the sequel as *separation conditions*.

Notice that if $i \leqslant j \leqslant \infty$, then $eS^{(j)}f \Rightarrow eS^{(i)}f$. Our goal is to investigate the meaning of the equivalence of some or all of the separation conditions. Before doing so, however, we pause to consider the two extreme conditions $S^{(1)}$ and $S^{(\infty)}$. In connection with this, it will prove convenient to follow F. Maeda's notation and write $e \nabla f$ in case $(e \vee x) \wedge f = x \wedge f$ for all $x \in L$. We then have

THEOREM 2.3 ([9], Theorem 2, p. 2). *Given $e, f \in L$, the following conditions are equivalent*:

(i) $eS^{(1)}f$;

(ii) $e \nabla f$;

(iii) $e \vee x = 1 \Rightarrow f \leqslant x$;

(iv) f *is contained in all complements of* e;

(v) $x = (x \vee e) \wedge (x \vee f)$ *for all* $x \in L$.

In view of this we shall be using the notation $eS^{(1)}f$ and $e \nabla f$ synonymously. Before looking at $S^{(\infty)}$ we need some additional terminology. If the set $\Theta(L)$ of congruence relations on L is partially ordered by the rule $\Theta_1 \leqslant \Theta_2$ iff $e\Theta_1 f \Rightarrow e\Theta_2 f$, it is well known that $\Theta(L)$ becomes a complete pseudo-complemented distributive lattice. Given $a, b \in L$, let $\Theta_{a,b}$ denote the smallest congruence relation identifying a and b, and for each congruence Θ, let Θ^* denote its psuedo-complement in $\Theta(L)$.

THEOREM 2.4. *Given $e, f \in L$, the following conditions are equivalent*:

(i) $eS^{(\infty)}f$;

(ii) $f \equiv O(\Theta^*_{e,0})$;

(iii) *there exists a congruence relation Θ such that $e \equiv O(\Theta)$ and $f \equiv O(\Theta^*)$*;

(iv) $e \approx g \Rightarrow fS^{(1)}g$;

(v) $e \approx g \Rightarrow f \wedge g = 0$.

Proof. (i) $\Rightarrow$ (ii). Let $x \leqslant f$ and $x \equiv O(\Theta_{e,0})$. Then by [10], Theorem 4.5, x is the join of a finite number of elements, each of which is projective to a subelement of e. Since $x \leqslant f$ and $eS^{(\infty)}f$, it is immediate that $x = 0$, so by [11], Theorem 4.13, p. 71, $f \equiv O(\Theta^*_{e,0})$.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (iv). Let $e \equiv O(\Theta)$, $f \equiv O(\Theta^*)$ and $e \approx g$. Then $g \equiv O(\Theta)$, so by [11], Theorem 4.13, p. 71, $gS^{(1)}f$.

(iv) $\Rightarrow$ (v). Clear.

(v) $\Rightarrow$ (i). Let $e_1 \leqslant e, f_1 \leqslant f$ and $e_1 \approx f_1$. By [10], Lemma 4.1, there exists an element $g \geqslant f_1$ with $e \approx g$. Then by (v), $f \wedge g = 0$ implies $f_1 = f_1 \wedge f \wedge g = 0$, thus establishing $eS^{(\infty)}f$.

We are now ready to begin our discussion of the various separation axioms. The next lemma serves to clarify the relation between $S^{(i)}$ and $S^{(i+1)}$.

LEMMA 2.5. *For $e, f \in L$, $eS^{(i+1)}f$ holds iff $eS^{(i)}g$ for all g perspective to f.*

Proof. Suppose first that $eS^{(i)}g$ for all $g \sim f$. Let $e_1 \leqslant e$, $f_1 \leqslant f$ and $e_1 \sim^{(i+1)} f_1$. There then exists an element g_1 such that $e_1 \sim^{(i)} g_1 \sim f_1$. Now $g_1 \sim f_1 \leqslant f$ implies (see [10], Lemma 4.1) that there exists an element $g \geqslant g_1$ such that $g \sim f$. By hypothesis, $eS^{(i)}g$, so we must have $e_1 = g_1 = 0$. It follows that $eS^{(i+1)}f$.

Suppose conversely that $eS^{(i+1)}f$ and let $f \sim g$. If $e_1 \leqslant e$, $g_1 \leqslant g$ and $e_1 \sim^{(i)} g_1$, then $g_1 \sim f_1$ for some $f_1 \leqslant f$, so $e_1 \sim^{(i+1)} f_1$ and $e_1 = f_1 = 0$. This leads immediately to

THEOREM 2.6. *If $S^{(i)} \Rightarrow S^{(i+1)}$, then $S^{(j)} = S^{(k)}$ for all $j, k \geqslant i$.*

Proof. It clearly suffices to show that if $S^{(i)} \Rightarrow S^{(i+1)}$, then $eS^{(i)}f \Rightarrow eS^{(\infty)}f$. Accordingly, let $e \in L$ be fixed, and say that f has property P in case $eS^{(i)}f$. Then if f has property P and $g \leqslant f$, clearly g has property P; moreover, by Lemma 2.5, any element perspective to f will have property P. It follows that P is a *perspective property* in the sense of [10]. Calling x a P-*element* when it has property P, we see ([10], Theorem 3.4) that the ideal J generated by the P-elements is the kernel of a congruence relation Θ. Suppose $x \leqslant e$ and $x \in J$. Then x is the join of a finite number of P-elements and $x \leqslant e$. But t a P-element implies $e \wedge t = 0$. It follows that $x = 0$ and so ([11], Theorem 4.13, p. 71) $e \equiv O(\Theta^*)$. By Theorem 2.4, $eS^{(i)}f \Rightarrow eS^{(\infty)}f$ as desired.

At this point the reader may very well ask when (if ever) one has $S^{(i)} \Rightarrow S^{(i+1)}$. It is therefore appropriate to consider some examples.

EXAMPLE 1. Let L be a complemented modular lattice. Let $eS^{(1)}f$, $0 < e_1 \leqslant e$, $0 < f_1 \leqslant f$ and $e_1 \approx f_1$. Then by [12], Lemma 9, p. 91, there exist e_2, f_2 such that $0 < e_2 \leqslant e_1$, $0 < f_2 \leqslant f_1$ and $e_2 \sim f_2$, a contradiction. It follows that $eS^{(\infty)}f$, so *all* of the separation conditions coincide.

EXAMPLE 2. Following the terminology of Holland, Jr. [7], we agree to call e and f *strongly perspective* if they have a common complement in their join. It is fairly easy to show that in the projection lattice of a Baer *-ring A, the following conditions are equivalent: (i) e and f are unrelated with respect to strong perspectivity; (ii) $eS^{(1)}f$; (iii) $eS^{(\infty)}f$;

(iv) $eAf = (0)$. The proof of this fact will be published elsewhere in a paper on Baer *-rings.

EXAMPLE 3. The lattice given in [7], p. 342, is relatively complemented, but $S^{(1)}$ does *not* imply $S^{(2)}$. (See Remark 3.6.)

Remark 2.7. Making use of Lemma 2.5, it is easy to show that $S^{(1)} \Rightarrow S^{(2)}$ is equivalent to every *normal* ideal (see [9]) being the kernel of a congruence relation, hence by [9], Theorem 8, p. 6, a central element of the completion of L by cuts.

3. The orthomodular case. An *orthomodular lattice* is a lattice L with 0 and 1 possessing a unary operation $a \to a'$ satisfying

$$a \wedge a' = 0, \quad a \vee a' = 1, \quad (a')' = a,$$

$$a \leqslant b \Rightarrow b' \leqslant a',$$

$$a \leqslant b \Rightarrow b = a \vee (b \wedge a').$$

These lattices have been studied in great detail in recent years. We refer the reader to [1], p. 52-3, [4] or [7] for an introduction to the subject. We merely remark that the projection lattice of a von Neumann algebra, any Boolean algebra, as well as any orthocomplemented modular lattice is orthomodular, and any orthomodular lattice is relatively complemented. *For the remainder of this section, L will denote an orthomodular lattice.*

Definition 3.1. Let $e \sim^{s} f$ denote the fact that e and f are *strongly perspective* in the sense that they are perspective in $[0, e \vee f]$, and let $eS^{(0)}f$ denote the fact that they are unrelated with respect to strong perspectivity.

In working with orthomodular lattices, the notion of *commutativity* is of vital importance. Basically, *e commutes* with f, denoted eCf, if and only if $e = (e \wedge f) \vee (e \wedge f')$. The elementary properties of commutativity were established independently by Foulis [4] and Holland [6]. Essentially, one has the following: (i) $e \leqslant f \Rightarrow eCf$; (ii) $eCf \Rightarrow fCe$; (iii) $eCf \Rightarrow e'Cf$; (iv) the set of elements commuting with e forms a sublattice of L closed under the formation of orthocomplements as well as any existing suprema or infima; (v) $eCf \Leftrightarrow (e \vee f') \wedge f = e \wedge f$; (vi) e is central if and only if eCf for all $f \in L$. The most important single fact about commutativity is contained in the next theorem due independently to Foulis ([4], Theorem 5, p. 68) and Holland ([6], Theorem 3, p. 69).

THEOREM 3.2 (Foulis-Holland). *If any two of the three relations eCf, fCg and eCg hold, then $(e \vee f) \wedge g = (e \wedge g) \vee (f \wedge g)$ and $(e \wedge f) \vee g = (e \vee g) \wedge (f \vee g)$.*

Following the terminology of Foulis [4], p. 66, we introduce the *Sasaki projection* $\varphi_f : L \to L$ by the formula $e\varphi_f = (e \vee f') \wedge f$ for all $e \in L$ and note that by [4], Lemmas 1 and 2, p. 66-67, we have: (i) $\varphi_f = \varphi_f\varphi_f$; (ii) $(e\varphi_f)'\varphi_f = e' \wedge f$; (iii) if $\bigvee_a e_a$ exists in L, then $\bigvee_a(e_a\varphi_f)$ exists and equals $(\bigvee_a e_a)\varphi_f$; (iv) $e\varphi_f = e$ iff $e \leqslant f$; (v) $e\varphi_f = 0$ iff $e \perp f$ in the sense that $e \leqslant f'$; (vi) $e\varphi_f = f$ iff $e \vee f' = 1$; (vii) eCf iff $\varphi_e\varphi_f = \varphi_f\varphi_e$.

Remark 3.3. It is an immediate consequence of the Foulis-Holland theorem (known both to Foulis and Holland) that every interval sublattice $[e, f] = \{x \in L : e \leqslant x \leqslant f\}$ is itself orthomodular with respect to the orthocomplementation

$$g^{\#} = (e \vee g') \wedge f = e \vee (g' \wedge f).$$

(See also [1], p. 53.) It is an easy matter to show that aCb iff b is the relative orthocomplement of a in $[a \wedge b, a \vee b]$. For if aCb, by the Foulis-Holland theorem,

$$\begin{aligned} a^{\#} &= [a' \vee (a \wedge b)] \wedge (a \vee b) \\ &= (a' \vee a) \wedge (a' \vee b) \wedge (a \vee b) \\ &= (b \vee a) \wedge (b \vee a') = b. \end{aligned}$$

On the other hand, if $b = (a \wedge b) \vee [(a \vee b) \wedge a']$, then $b \vee a' = (a \wedge b) \vee \vee a'$ and $(b \vee a') \wedge a = [(a \wedge b) \vee a'] \wedge a = a \wedge b$, thereby showing bCa.

Recalling that in an arbitrary relatively complemented lattice with 0 and 1, $eS^{(1)}f \Leftrightarrow e\nabla f$, we begin our discussion of separation conditions by investigating the ∇-relation in the orthomodular lattice L.

THEOREM 3.4. *Given $e, f \in L$, the following conditions are equivalent*:

(i) $e\nabla f$;

(ii) $g \wedge e' = 0 \Rightarrow f \perp g$;

(iii) $f \perp e\varphi_g$ *for all* $g \in L$;

(iv) $\varphi_e\varphi_g\varphi_f = 0$ *for all* $g \in L$.

Proof. (i) $\Rightarrow$ (ii). If $g \wedge e' = 0$, then $g' \vee e = 1$, so by Theorem 2.3, $f \leqslant g'$.

(ii) $\Rightarrow$ (iii). $e\varphi_g \wedge e' = (e \vee g') \wedge g \wedge e' = 0$ implies $f \perp e\varphi_g$.

(iii) $\Rightarrow$ (iv). By [4], Lemma 1, p. 66, $f \perp e\varphi_g$ iff $e\varphi_g\varphi_f = 0$ and this clearly implies $\varphi_e\varphi_g\varphi_f = 0$.

(iv) $\Rightarrow$ (i). Let x be a complement of e. Then $0 = e\varphi_x\varphi_f = [(e \vee x) \wedge \wedge x']\varphi_f = x'\varphi_f$ implies $x' \leqslant f'$, so $f \leqslant x$. By Theorem 2.3, $e\nabla f$.

Remark 3.5. For those having some knowledge of Baer *-semigroups (see [3]), the above theorem states that for two closed projections e and f, $e\nabla f$ in $P'(S)$ if and only if $egf = 0$ for all $g \in P'(S)$.

Remark 3.6. Consider the following separation conditions (listed in order of implication):

(i) $eS^{(2)}f$;

(ii) $f\nabla e\varphi_x$ for all $x \in L$;

(iii) $e \sim g \Rightarrow f \perp g$;

(iv) $e\nabla f$.

In the example shown in [7], Fig. 1, p. 342, $a\nabla g$, $g \sim b$, but a is not orthogonal to b. Thus a and g satisfy (iv) but not (iii). It is not known which of the other implications is strict.

We are now ready to have a look at the separation condition $S^{(0)}$ on L. Our approach will be via the theory of Sasaki projections.

LEMMA 3.7. *Let $a \leqslant e \wedge f \leqslant e \vee f \leqslant b$. Then if $f^{\#}$ is the relative orthocomplement of f in $[a, b]$, $e\varphi_f = (e \vee f^{\#}) \wedge f$.*

Proof. Making use of the Foulis-Holland theorem we may write

$$\begin{aligned}(e \vee f^{\#}) \wedge f &= [e \vee a \vee (f' \wedge b)] \wedge f = [e \vee (f' \wedge b)] \wedge f \\ &= (e \vee f') \wedge (e \vee b) \wedge f = (e \vee f') \wedge b \wedge f \\ &= (e \vee f') \wedge f.\end{aligned}$$

LEMMA 3.8. *For arbitrary $e, f \in L$, $e\varphi_f$ and $f\varphi_e$ are perspective in the interval from their meet to their join.*

Proof. Let $g = e\varphi_f$ and $h = f\varphi_e$. By [4], Corollary to Theorem 5, p. 68, g and h' are complements in L. Notice that $g\varphi_h = (g \vee h') \wedge h = h = f\varphi_e$ while $h\varphi_g = (h \vee g') \wedge g = g = e\varphi_f$. Letting $x^{\#}$ denote the relative orthocomplement of x in $[g \wedge h, g \vee h]$ we infer by Lemma 3.6 that $g\varphi_h = (g \vee h^{\#}) \wedge h$, $h\varphi_g = (h \vee g^{\#}) \wedge g$. Applying [4], Corollary to Theorem 5, p. 68, to the interval $[h \wedge g, h \vee g]$ we see that $g\varphi_h$, $h\varphi_g$ are perspective in that interval; i.e., $e\varphi_f$ and $f\varphi_e$ are perspective in the interval $[g \wedge h, g \vee h]$ as desired.

LEMMA 3.9. *If $x \wedge e = x \wedge e' = 0$, then $x\varphi_e \sim^s x\varphi_{e'}$.*

Proof. Repeated application of the Foulis-Holland theorem will produce the fact that

$$x \vee x\varphi_e = x \vee x\varphi_{e'} = x\varphi_e \vee x\varphi_{e'} = (x \vee e) \wedge (x \vee e').$$

We are now able to state the following result, due essentially to S. S. Holland, Jr. Its importance stems from [7], Corollary 2, p. 340.

THEOREM 3.10. *Let $e, f \in L$. Then $eS^{(0)}f$ if and only if $e \wedge f = 0$ and e is central in $[0, e \vee f]$.*

Proof. Let $eS^{(0)}f$. By Lemma 3.8 we must have $e\varphi_f = 0$, so $e \perp f$. Dropping down to $[0, e \vee f]$ we may as well assume $f = e'$. Then for arbitrary x, $x \geqslant (x \wedge e) \vee (x \wedge e')$ and if y is a complement of $(x \wedge e) \vee$

$\vee (x \wedge e')$ in $[0, x]$, clearly $y \wedge e = y \wedge e' = 0$. By Lemma 3.9, $y\varphi_e \sim^s y\varphi_{e'}$, so $y\varphi_e = y\varphi_{e'} = 0$. But this implies $y \leqslant e \wedge e' = 0$, so $x = (x \wedge e) \vee (x \wedge e')$. This shows that xCe for all x, so e is indeed central in $[0, e \vee f]$. The converse is clear.

Making use of the above theorem and the fact that $eS^{(0)}f$ is equivalent to $e \nabla f$ in the interval from 0 to $e \vee f$, it is possible to make quite an extensive list of conditions all equivalent to $S^{(0)}$. Since the conditions are due as much to Foulis and Holland as to the author, and since they have become part of the "folklore" of orthomodular lattice theory, it seems inappropriate to reproduce them here. Rather, we refer the interested reader to Foulis [5]. One can, however, also characterize the condition $S^{(0)}$ in terms of Sasaki projections as follows:

THEOREM 3.11. *Let $e, f \in L$ with $e \wedge f = 0$. Then $eS^{(0)}f$ holds iff $x\varphi_e \vee x\varphi_f = x\varphi_{e \vee f}$ for all $x \in L$.*

Proof. Assume first that $x\varphi_e \vee x\varphi_f = x\varphi_{e \vee f}$ for all $x \in L$. Let $x \leqslant e \vee f$ and choose g to be a complement of $(x \wedge e) \vee (x \wedge f)$ in $[0, x]$. Then $g \wedge e = g \wedge f = 0$, so $g'\varphi_e = e$ and $g'\varphi_f = f$. Hence $g'\varphi_{e \vee f} = g'\varphi_e \vee g'\varphi_f = e \vee f$, so $g' \vee (e' \wedge f') = 1$ and $g = g \wedge (e \vee f) = 0$. It follows that $x = (x \wedge e) \vee (x \wedge f)$ for all $x \leqslant e \vee f$ and, consequently, that e is central in $[0, e \vee f]$. By Theorem 3.10, $eS^{(0)}f$.

Suppose conversely that $eS^{(0)}f$. Then by Theorem 3.10, e is central in $[0, e \vee f]$. Routine computation shows that $x\varphi_e = x\varphi_{e \vee f}\varphi_e = (x\varphi_{e \vee f}) \wedge e = [x \vee (e' \wedge f')] \wedge e$. Similarly, $x\varphi_f = [x \vee (e' \wedge f')] \wedge f$. Now $x' \wedge (e \vee f) \leqslant e \vee f$ implies that $eC[x' \wedge (e \vee f)]$, so $eC[x \vee (e' \wedge f')]$. By the Foulis-Holland theorem,

$$\begin{aligned} x\varphi_e \vee x\varphi_f &= \{[x \vee (e' \wedge f')] \wedge e\} \vee \{[x \vee (e' \wedge f')] \wedge f\} \\ &= [x \vee (e' \wedge f')] \wedge (e \vee f) = x\varphi_{e \vee f}. \end{aligned}$$

We now investigate more closely the relation between $S^{(0)}$ and $S^{(1)}$, and what, precisely, it means for them to coincide.

LEMMA 3.12. *$e \nabla f$ implies $e\varphi_g S^{(0)} f\varphi_g$ for all $g \in L$.*

Proof. Given $x \geqslant g'$, by Theorem 2.3,

$$x = (x \vee e) \wedge (x \vee f) = (x \vee g' \vee e) \wedge (x \vee g' \vee f).$$

Applying Theorem 2.3 to the interval $[g', 1]$, we see that $g' \vee e \nabla g' \vee f$ in that interval. We now make use of the fact that $a \to a \wedge g$ is an isomorphism of $[g', 1]$ onto $[0, g]$ to conclude that $e\varphi_g \nabla f\varphi_g$ in $[0, g]$, whence $e\varphi_g S^{(0)} f\varphi_g$ in L.

LEMMA 3.13. *$e \nabla f$ is equivalent to $eS^{(0)}f\varphi_g$ for all $g \in L$.*

Proof. Let $e\nabla f$. Fix $g\in L$ and set $h = (f' \wedge g') \vee g = (f\varphi_{g'})'$. By Theorem 3.4, $e \leqslant h$, so $e = e\varphi_h$. Notice, however, that

$$\begin{aligned} f\varphi_h &= (f \vee h') \wedge h = (f \vee f\varphi_{g'}) \wedge h = (f \vee g) \wedge (f \vee g') \wedge h \\ &= [(f \vee g) \wedge (f \vee g') \wedge (f' \wedge g')] \vee [(f \vee g) \wedge (f \vee g') \wedge g] \\ &= 0 \vee [(f \vee g') \wedge g] = f\varphi_g. \end{aligned}$$

By Lemma 3.12, $eS^{(0)}f\varphi_g$. The converse follows immediately from Theorem 3.4.

LEMMA 3.14. *Let R be a symmetric relation on L such that*: (i) eRf, $e_1 \leqslant e \Rightarrow e_1Rf$; (ii) $eRf \Rightarrow e \wedge f = 0$; (iii) $eRf \Rightarrow eRf\varphi_g$ *for all* $g\in L$. *Then* $eRf \Rightarrow eS^{(\infty)}f$.

Proof. Let $e\in L$ be fixed, and say that f has property P in case eRf. Note that if eRf and $f \sim g$ with common complement x, then $f\varphi_{x'} = x'$ and $x'\varphi_g = g$ shows eRg. It follows that P is a perspective property, so as in the proof of Theorem 2.6, the ideal generated by the P-elements is the kernel of a congruence relation Θ such that $e \equiv O(\Theta^*)$. It follows as in 2.6 that $eRf \Rightarrow eS^{(\infty)}f$.

Combining the above two lemmas we now have

THEOREM 3.15. *If $eS^{(0)}f \Rightarrow eS^{(1)}f$, then the separation conditions $S^{(0)}$, $S^{(1)}$, $S^{(2)}, \ldots, S^{(\infty)}$ all coincide.*

It seems worth mentioning at this point that if L is modular, then perspectivity and strong perspectivity coincide. It is immediate that $S^{(0)} \Rightarrow S^{(1)}$, so by Theorem 3.15, the separation conditions are all equivalent.

4. The relative center property. *In this section L will once again denote an orthomodular lattice.*

We agree to say that L has the *relative center property* (RCP) if e central in $[0, a]$ implies $e = z \wedge a$ with z central in L. As was pointed out in section 1, the reason for considering this property is that it is valid in any complete complemented modular lattice, as well as in the projection lattice of a von Neumann algebra. If one is ever to abstractly characterize these lattices, some knowledge of the relative center property would appear useful. Recalling that a lattice with 0 and 1 is *irreducible* if its center consists of 0 and 1, we first have

THEOREM 4.1. *The following conditions are equivalent*:

(i) *L is irreducible and has the relative center property*;

(ii) *every interval $[0, a]$ is irreducible*;

(iii) *every interval $[e, f]$ is irreducible.*

We omit the easy proof but refer the interested reader to Catlin [2] for a discussion of irreducibility conditions on an orthomodular lattice. For the general case we have.

LEMMA 4.2. *Let L have the relative center property. The following are equivalent*:

(i) $eS^{(0)}f$;

(ii) $eS^{(1)}f$;

(iii) *there exists a central element z with $e \leqslant z$ and $f \leqslant z'$.*

Proof. (i) $\Rightarrow$ (iii). If $eS^{(0)}f$, then e is central in $[0, e \vee f]$, so by RCP there exists a central element z such that $e = (e \vee f) \wedge z = e \vee (f \wedge z)$. Then $f \wedge z \leqslant e$ implies $f \wedge z = 0$, so $e \leqslant z$ and $f \leqslant z'$.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Clear.

COROLLARY 4.3. *If L has the relative center property, then the separation conditions $S^{(0)}$, $S^{(1)}$, $S^{(2)}$, ..., $S^{(\infty)}$ all coincide.*

Now let L be complete, and for each $e \in L$, $\gamma(e) = \bigwedge \{z \in L : z$ central, $z \geqslant e\}$. The element $\gamma(e)$, usually called the *central cover* of e, is evidently the smallest central element dominating e. The next theorem tells what it means for L to have the relative center property.

THEOREM 4.4. *For a complete orthomodular lattice L, the following are all equivalent*:

(i) *L has the relative center property*;

(ii) $eS^{(0)}f \Rightarrow eS^{(1)}f$;

(iii) $eS^{(0)}f \Rightarrow \gamma(e) \wedge \gamma(f) = 0$.

Proof. (i) $\Rightarrow$ (ii) by Lemma 4.2.

(ii) $\Rightarrow$ (iii). By Theorem 3.15 the separation conditions $S^{(0)}, S^{(1)}, \ldots, S^{(\infty)}$ all coincide. Given $e \in L$, let $e^{\nabla} = \bigvee \{f \in L : e \nabla f\}$. By Theorem 2.3, $e \nabla e^{\nabla}$, and as in the proof of Theorem 2.6, there exists a congruence relation Θ on L whose kernel is $[0, e^{\nabla}]$. It is immediate that e^{∇} is central. We now note that $eS^{(0)}f \Rightarrow e \nabla f \Rightarrow f \leqslant e^{\nabla}$ and $e \wedge e^{\nabla} = 0 \Rightarrow e \leqslant (e^{\nabla})'$. Since e^{∇} and $(e^{\nabla})'$ are central, we see that $\gamma(e) \leqslant (e^{\nabla})'$ and $\gamma(f) \leqslant e^{\nabla}$, so $\gamma(e) \wedge \gamma(f) = 0$.

(iii) $\Rightarrow$ (i). Let e be central in $[0, a]$. Then $eS^{(0)} a \wedge e' \Rightarrow \gamma(e) \wedge \gamma(a \wedge e') = 0$. Hence $a \wedge \gamma(e) = \big(e \vee (a \wedge e')\big) \wedge \gamma(e) = e \vee \big(a \wedge e' \wedge \gamma(e)\big) = e$.

Although the assumption of completeness in Theorem 4.4 can be considerably weakened, we present herewith an example to show that it cannot be entirely removed. Let $\mathfrak{H}$ denote an infinite-dimensional Hilbert space and L the set of ordered pairs $(\mathfrak{M}, \mathfrak{N})$ of closed subspaces such that both $\mathfrak{M}$ and $\mathfrak{N}$ are either finite-dimensional or cofinite-dimensional. If L is partially ordered in the obvious manner by $(\mathfrak{M}_1, \mathfrak{N}_1) \leqslant (\mathfrak{M}_2, \mathfrak{N}_2)$ iff $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ and $\mathfrak{N}_1 \subseteq \mathfrak{N}_2$, it is easy to show that L becomes an irreducible orthocomplemented modular lattice. Yet if $\mathfrak{M}$ is finite-dimensional, $(\mathfrak{M}, 0)$ is central in $[(0, 0)\ (\mathfrak{M}, \mathfrak{M})]$. This shows that an irreducible orthocomplemented modular lattice may have reducible intervals. In view of Theorem 4.1, we conclude that the relative center property may not be deduced from the fact that $eS^{(0)}f \Rightarrow eS^{(1)}f$.

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