COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

AN EXAMPLE OF THE DECOMPOSABILITY SEMIGROUP

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The concept of the decomposability semigroup associated with probability measures has been introduced by Urbanik in [2]. It has also been proved there that some probabilistic properties of measures correspond to algebraic and topological properties of their decomposability semigroups. Some non-trivial examples of decomposability semigroups for probability measures on the real line have been given in [3]. Even on the real line the problem of characterization of those semigroups which are decomposability semigroups for probability measures is still open.

Let P be a probability measure on the real line R. For every $c \in R$ we shall denote by T_c the mapping $T_c x = cx$ $(x \in R)$. Further, $T_c P$ will denote the measure defined by the formula $T_c P(E) = P(T_c^{-1}(E))$ for all Borel subsets E of R. The decomposability semigroup $D(P)$ corresponding to P consists of all real numbers c for which there exists a probability measure P_c such that $P = T_c P * P_c$. The semigroup operation is simply the multiplication of numbers.

Let $\hat{P}$ be the characteristic function of P, i.e., the Fourier transform of P,

$$\hat{P}(t) = \int_{-\infty}^{\infty} e^{itx} P(dx).$$

It is evident that $c \in D(P)$ if and only if there exists a probability measure P_c such that

$$\hat{P}(t) = \hat{P}(ct)\hat{P}_c(t) \qquad (t \in R).$$

It is well known that P is non-degenerate if and only if $D(P)$ is compact (see [2], Theorem 1). In other words, for non-degenerate P, $D(P)$ is a compact subsemigroup of the multiplicative semigroup $[-1, 1]$. Further, it is clear that $0 \in D(P)$ and $1 \in D(P)$. Urbanik raised the problem whether this condition characterizes decomposability semigroups among compact ones. It is easy to prove that $-1 \in D(P)$ if and only if P is a translate of a symmetric probability measure. The theory of self-decomposable probability measures affords important examples of decomposability semigroups. Namely, Lévy's characterization of non-degenerate self-decomposable laws P is equivalent to the inclusion $[0, 1] \subset D(P)$ (see [1], Section 23.3). Hence, in particular, we get the following statement: a non-degenerate probability measure P is a translate of a symmetric self-decomposable one if and only if $D(P) = [-1, 1]$.

Urbanik raised also the following problem: Let P be a symmetric probability measure such that its decomposability semigroup contains a neighbourhood of the origin. Is it self-decomposable? In other words, is the equality $D(P) = [-1, 1]$ true? The aim of the present note is to love a negative answer to this problem. Namely, we shall prove the folgiwing

THEOREM. *There exists a probability measure Q such that $D(Q)$ contains a neighbourhood of the origin and Q is not self-decomposable.*

Proof. The function $f(t) = \max(1-|t|, 0)$ is the characteristic function of a symmetric probability measure (see [1], p. 218). Since the infinite product $\prod_{k=0}^{\infty} f(t/2^k)$ is convergent uniformly on every compact set, we infer that it is the characteristic function of a symmetric probability measure which will be denoted by Q. Since $\hat{Q}(t) = 0$ for $|t| \geqslant 1$, the probability measure Q is not self-decomposable. Given $c \in [-\frac{1}{2}, \frac{1}{2}]$, we put $f_c(t) = \hat{Q}(t)/\hat{Q}(ct)$ if $|t| < 1$ and $f_c(t) = 0$ otherwise. It is obvious that the function f_c is even, $f_c(0) = 1$, and $f_c(t) = 0$ for $|t| \geqslant 1$. To prove that f_c is the characteristic function of a probability measure it suffices to show that it is convex from below on the interval $[0, 1]$ (see [1], p. 217). By a simple calculation for $t \in [0, 1]$ we get the formula

$$\frac{d^2 f_c(t)}{dt^2} = f_c(t) g_c(t),$$

where

$$g_c(t) = \left(\sum_{k=0}^{\infty} \frac{1}{2^k - t} - \sum_{k=0}^{\infty} \frac{|c|}{2^k - |c|t}\right)^2 - \sum_{k=0}^{\infty} \frac{1}{(2^k-t)^2} + \sum_{k=0}^{\infty} \frac{c^2}{(2^k - |c|t)^2}.$$

For $k = 0, 1, \ldots$ put

$$a_k(t) = \frac{1}{2^{k+1} - t} - \frac{|c|}{2^k - |c|t}$$

and

$$b_k(t) = \frac{1}{2^{k+1} - t} + \frac{|c|}{2^k - |c|t}.$$

Since $|c| \in [0, \frac{1}{2}]$ and $t \in [0, 1]$, we have the inequalities

$$a_k(t) \geqslant 0 \quad \text{and} \quad b_k(t) \leqslant 2 \quad (k = 0, 1, \ldots).$$

Thus

$$g_c(t) = \Big(\sum_{k=0}^{\infty} a_k(t)\Big)^2 + \frac{2}{1-t} \sum_{k=0}^{\infty} a_k(t) - \sum_{k=0}^{\infty} a_k(t) b_k(t) \geqslant 0,$$

which implies the inequality

$$\frac{d^2}{dt^2} f_c(t) \geqslant 0 \quad \text{in } [0, 1].$$

Consequently, f_c is convex from below in $[0, 1]$. Hence it follows that $[-\frac{1}{2}, \frac{1}{2}] \subset D(Q)$, which completes the proof.

REFERENCES

[1] M. Loève, *Probability theory*, New York 1950.

[2] K. Urbanik, *Operator semigroups associated with probability measures*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 23 (1975), p. 75-76.

[3] — *Some examples of decomposability semigroups*, ibidem 24 (1976), p. 915-918.

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Reçu par la Rédaction le 29. 4. 1976