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Stanisław Knapowski

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Stanisław Knapowski was born in Poznań to a lawyer, later Professor at Poznań University, Roch Knapowski and Zofia née Krysiewicz. Until the outbreak of the Second World War he lived with parents in his native town, but soon, like thousands of other inhabitants, they were expelled from there by German occupants and settled in the Kielce region. After the war the family has returned to Poznań. There in 1949 Stanisław graduated from gymnasium and entered Poznań University to study mathematics. Three years later he moved to Wrocław to continue his studies at Wrocław University. While he was still a student, he was appointed to the post of an assistant. After receiving in 1954 the degree of M.A. he became an assistant at Poznań University.

In 1956 he was introduced to Professor Turán and began to work under his guidance, what has a dominant influence upon his all scientific activity. They published 18 papers of joint authorship. In 1957 Stanisław Knapowski took his doctor's degree on the ground of a paper on some applications of Turán's methods in the analytical number theory. Then he stayed for 10 months in Cambridge, England, where he worked under L. J. Mordell, took a part in a seminar of J. W. S. Cassels, and heard A. E. Ingham's lectures. Afterwards he visited some other universities in Great Britain, Belgium, France, and Holland.

In 1960 he received docent's degree at the University of Adam Mickiewicz in Poznań on the ground of the paper *On new »explicit formulas« in prime number theory* (cf. [11] and [15] in the list of his scientific papers, p. 318-321). In 1962 he was awarded Mazurkiewicz Prize by the Polish Mathematical Society. All the academic year 1962/63 he lectured at Tulane University, New Orleans (USA).

Since the second half of 1964 he stayed in the West, where he lectured at several universities, among others in Marburg (W. Germany), Gainesville and Miami (USA). Sloan Fundation granted him a scholarship to continue his scientific investigations in various mathematical centres.

Stanisław Knapowski was a member of the Polish Mathematical Society and American Mathematical Society.

He was killed in a car accident in Florida.

His mathematical activity is described by Professor Turán in the next article.

COMMEMORATION ON STANISŁAW KNAPOWSKI

BY

PAUL TURÁN (BUDAPEST)

The last letter I got from Knapowski was dated from September 13, 1967. When reading its mathematical content some days later I had not the faintest idea that its sender is no more at that time alive. He apparently lost control over his car when returning from the airport in Miami on September 28, 1967, and died being only 36 years old.

Our connection — apart from a few letters before — started actually in September of 1956 when I had a series of lectures on various Polish universities starting in Lublin, dealing mainly with a new analytical method. He came to Lublin and in several conversations I realised his quick and deep understanding. We spoke much on possible applications of the method in the analytical number-theory and in 1958 appeared his first paper in this direction (see [8]) (1). It is shown in it, roughly speaking, that denoting as usually the number of primes not exceeding x and $\equiv l \bmod k$ by $\pi(x, k, l)$ the good approximability of $\pi(x, k, 1)$ by

$$F_k(x) = \frac{1}{\varphi(k)} \int_2^x \frac{dv}{\log v} \tag{1}$$

implies "almost as good an approximation" of $\pi(x, k, l)$ by $F_k(x)$. The interest of this theorem is partly due to the fact that the case $l = 1$ seems to be easier than the case of any other residue classes $\bmod k$ inasmuch as Dirichlet's theorem can be proved for $l = 1$ "really arithmetically directly and shortly" which is not the case for the other residue classes $\bmod k$ (including the case $l = -1$). More exactly, he proved [8] that with the notation

$$\Delta(x, k, l) = \pi(x, k, l) - F_k(x) \tag{2}$$

(1) The numbers in brackets refer to the list of scientific papers of Stanisław Knapowski which follows.

we have for (2)

(3) $$T > \max(c, e^{k^2})$$

the inequality

(4) $$\max_{1\leqslant x\leqslant T} |\Delta(x, k, l)| \leqslant T^{\delta(T)} \exp\left(4\frac{\log T}{\sqrt{\log\log T}}\right)\{\sqrt{T} + \max_{1\leqslant x\leqslant T} |\Delta(x, k, 1)|\}.$$

Here the meaning of $\delta(T)$ is the following:

If $\varrho = \beta + i\gamma$ stand for the nontrivial roots of all Dirichlet $L(s, k, \chi)$ functions belonging to the modulus k and

$$\varepsilon(T) = \max_{|\gamma|\leqslant T} \beta,$$

then $\delta(T)$ is defined by

$$\delta(T) = \varepsilon(\sqrt{T}) - \varepsilon\big(\exp(\sqrt{\log\log T})\big).$$

Obviously, $\delta(T) \to 0$ with $\frac{1}{T}$, and if the Riemann-Piltz conjecture is true then $\delta(T) \equiv 0$, of course. The main tool in the proof was what I called the second main theorem of the mentioned method; this asserts that if $b_1, b_2, \ldots, b_n$ are arbitrary complex numbers, $z_1, z_2, \ldots, z_n$ are complex numbers with

(5) $$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|$$

and m is an arbitrary positive integer, then for a suitable integer ν_0 with $m+1 \leqslant \nu_0 \leqslant m+n$ holds the inequality

(6) $$|b_1 z_1^{\nu_0} + b_2 z_2^{\nu_0} + \ldots + b_n z_n^{\nu_0}| \geqslant \left(\frac{n}{8e(m+n)}\right)^n \min_{j=1,\ldots,n} |b_1 + \ldots + b_j|.$$

The meaning of this theorem is roughly that by a suitable choice of the exponent ν_0 it permits the *lower* estimation of a sum essentially by its maximal term. This is particularly clear in the important case when all b_j's are 1, but in many other not less important instances it is not so; the main trouble is with the factor

(7) $$\min_{j=1,\ldots,n} |b_1 + \ldots + b_j|.$$

In the cases (see [12]) where all $\mathrm{Re}\, b_j$ happened to be nonnegative there were no difficulties (after such cases were found at all), but this occurs seldom. In his papers [16], [22] and [24] Knapowski found an ingenious and usable remedy for some difficult and interesting cases by

(2) c mean throughout this paper unspecified explicitely calculable positive numerical constants (which are of course independent of k).

restricting considerably the range of j in (7) at the cost of worsening "not too much" the lower bound. Instead of giving the exact details of his rather technical lemma I from [16] we show its usefulness by quoting some theorems that he proved using it (they were stronger than those annouced by me in 1959 at the sixth Mathematical Congress of the Italian Mathematical Society in Naples).

THEOREM I. *If* (³)

$$T > \max(c, e^{k^{20000}}), \tag{8}$$

then, with the abbreviation

$$X = T\exp\{-(\log T)^{0,9}\},$$

for all $(l_1, k) = (l_2, k) = 1, l_1 \neq l_2$, *holds the inequality*

$$\int_X^T \frac{|\Pi(x, k, l_1) - \Pi(x, k, l_2)|}{x}\, dx > T^{1|4}.$$

Here $\Pi(x, k, l)$ means the "Riemann function"

$$\sum_{\substack{p,a \\ p^a \equiv l \bmod k \\ p^a \leqslant x}} \frac{1}{a}.$$

Supposing further that no $L(s, k, \chi)$ functions vanish for

$$0 < \sigma < 1, \quad |t| \leqslant \max(c, k^7), \quad s = \sigma + it, \tag{9}$$

he proved the following theorem:

THEOREM II. *If* $T > \max(c, e^{e^k})$, *then, for all* $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$, (9) *implies the inequality*

$$\int_{X_1}^T \frac{|\pi(x, k, l) - \pi(x, k, l_2)|}{x}\, dx > \sqrt{T}\exp\left(-7\frac{\log T}{\log\log T}\right),$$

where

$$X_1 = T\exp\{-(\log T)^{3/4}\}.$$

In order to proceed to another important result of Knapowski we have to go back to Riemann. In his famous memoir from 1859 he risked the assertion

$$\Delta(x) \overset{\text{def}}{=} \pi(x) - \int_2^x \frac{dv}{\log v} < 0 \quad \text{for all } x > 2,$$

where $\pi(x)$ stands for the number of primes not exceeding x. This was

(³) In his paper he had k^{30L_0} instead of k^{20000} in (8); L_0 is the Linnik constant, i. e. a constant such that for all $(l, k) = 1$ there is a prime $< ck^{L_0}$ which $\equiv l \bmod k$. Recently, M. Jutila proved that $L_0 < 550$.

disproved by Littlewood in 1914 (in the same paper in which he disproved also the conjecture that for $x > x_0$ the function

$$\pi(x, 4, 1) - \pi(x, 4, 3) \tag{10}$$

does not change sign). Curiously enough his proof did not furnish any explicit upper bound for the smallest $x = x_1$ with $\Delta(x_1) > 0$; it has been done by Skewes only in 1955, but even this improved version could give no lower bound for the number $V(T)$ of sign changes of $\Delta(x)$ for $0 < x \leqslant T$. Long before I have suspected that a one-sided theorem of type (5)-(6) should solve the difficulty. The ideal type of such a theorem had been if (5) would imply the existence of integers ν_1 and ν_2 with $m+1 \leqslant \nu_1, \nu_2 \leqslant m+n$ such that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant g(n, m)|b_1 + \ldots + b_n|$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -g(n, m)|b_1 + \ldots + b_n|$$

with a positive $g(n, m)$; the trivial example $z_1 = z_2 = \ldots = z_n = 1$ shows, however, that no such theorem can be true. Nevertheless in 1959 I discovered that adding to (5) the natural and simple restriction

$$\varkappa \leqslant \min|\arg z_\nu| \leqslant \pi, \quad 0 < \varkappa \leqslant \frac{\pi}{2}, \ \nu = 1, 2, \ldots, n, \tag{11}$$

saves the situation; more exactly, (5) and (11) imply the existence of integers ν_1 and ν_2 with

$$m+1 \leqslant \nu_1, \quad \nu_2 \leqslant m + \left(3 + \frac{\pi}{\varkappa}\right) n, \tag{12}$$

so that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant \frac{1}{2n+1} \left(\frac{n}{24e^3\big(m+n(3+\pi/\varkappa)\big)}\right)^{2n} \min_{\mu=1,\ldots,n} \left|\operatorname{Re} \sum_{j=1}^{\mu} b_j\right|, \tag{13}$$

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -\frac{1}{2n+1} \left(\frac{n}{24e^3\big(m+n(3+\pi/\varkappa)\big)}\right)^{2n} \min_{\mu=1,\ldots,n} \left|\operatorname{Re} \sum_{j=1}^{\mu} b_j\right|. \tag{14}$$

I mentioned this theorem to Knapowski in a letter but I did not see at all how one could ascertain that (11) is fulfilled. By an ingenious idea (see his lemma in [17]) he settled the missing step and proved in his paper [19] the following theorem which was the first unconditional result concerning the number $V(T)$ of sign changes.

THEOREM III. *If* (4) $T > e_5(25)$, *then the inequality*

$$V(T) > e^{-25}\log_4 T$$

holds.

It was not difficult to realise that a combination of the "one-sided" theorem (12)-(13)-(14) with Knapowski's lemma in [18] has much significance also for the comparison of the distribution of rational primes in different residue classes $\bmod k$ (and even for two such residue classes $\bmod k_1$ and $\bmod k_2$, if only $\varphi(k_1) = \varphi(k_2)$). This was the starting point of a long sequence of joint papers (see [25]-[27], [30]-[34], [37], [38], [43]--[49], [52], [53])(5). The fact that Knapowski's lemma was replaced in [37] by a general lemma with shorter proof does not deduce a bit from the merit and promoting effect of his original lemma in the course of our work. All these comparison problems grew out from the assertion of Čebyšev in 1853 according to whom, in some contrast to Dirichlet's theorem, there should be

$$\lim_{n\to\infty}\sum_{p>2}(-1)^{(p-1)/2}e^{-p/x} = -\infty; \tag{15}$$

whence it would follow that in this sense there are more primes $\equiv 3 \bmod 4$ than $\equiv 1 \bmod 4$. The depth of these problems is indicated by the discovery of Hardy-Littlewood and Landau during the first world war that the truth of (15) is equivalent (6) to the assertion

$$L(s, 4, \chi_1) \neq 0 \quad \text{for } \sigma > \tfrac{1}{2}, \tag{16}$$

where $\chi_1(n, 4)$ is the non-principal character belonging to the modulus 4, and also, by (10), which shows that this preponderance of primes $\equiv 3 \bmod 4$ in *direct* sense is certainly false (which is certainly not a sign for strengthening the belief in the truth of (16)). These facts give an obvious interest on the one hand to extend this connection to general moduli and on the other hand to get insight into the oscillatory nature of $\pi(x, k, l_1) - \pi(x, k, l_2)$ or other analogous expressions which are *explicit in x and k*; e.g., such second type results would enable one to get upper bound depending explicitely on k for the first sign change of these expressions (if there is any). Until 1960 no such results were known though in a very weak form it was known to Landau that the number of solutions of the congruences $x^2 \equiv l_\nu \bmod k$ as well as the existence of *real* roots of the $L(s, k, \chi)$

(4) In what follows $e(x) = e_1(x)$ stands for e^x. Also $e_{\nu+1}(x) = e_1(e_\nu(x))$ and $\log_{\nu+1}x = \log_\nu(\log x)$ with $\log_1 x = \log x$.

(5) Since the last two papers are not yet actually submitted for publication, I shall indicate their content later in this note.

(6) As shown by Hardy and Littlewood, the same conclusion holds if (15) is replaced by

$$\lim_{x\to+\infty}\sum_{p>2}(-1)^{(p-1)/2}\log p\, e^{-p/x} = -\infty. \tag{17}$$

functions play a role in these questions. All of our results refer to "good" moduli k, i.e. to those for which no $L(s, k, \chi)$-functions have positive roots; even they are nonvanishing in a parallelogram of the form

$$\sigma \geqslant \tfrac{1}{2}, \quad |t| \leqslant E = E(k), \tag{18}$$

which can be assumed to satisfy

$$E(k) \leqslant \frac{\sqrt{\log k}}{k}. \tag{19}$$

As to the problems of the first kind, though it is not at all clear that a theorem of type (15) is true for all moduli we have found, at least for good k-moduli, that the truth of the relation

$$\lim_{x\to+\infty}\Big\{\sum_{p\equiv 1 \bmod k} \log p \exp\Big(-a\frac{E(k)^2}{\log k}\log^2\frac{p}{x}\Big) - \sum_{p\equiv l \bmod k} \log p \exp\Big(-a\frac{E(k)^2}{\log k}\log^2\frac{p}{x}\Big)\Big\} = -\infty \tag{20}$$

for appropriate positive numerical a and for all quadratic nonresidue l is equivalent to the truth of the Riemann-Piltz conjecture for all $L(s, k, \chi)$, $\chi \neq \chi_0$, functions (with the same fixed k as in (20))(⁷). As to the problems of the second kind let me mention only the theorem (see [34]) for the function

$$\psi(x, k, l) = \sum_{\substack{p,a \\ p^a \leqslant x,\, p^a \equiv l \bmod k}} \log p, \tag{21}$$

usual in the theory of primes, according to which all functions

$$\psi(x, k, l_1) - \psi(x, k, l_2), \quad (l_1, k) = (l_2, k) = 1,\ l_1 \neq l_2 \tag{22}$$

change sign in the interval

$$1 \leqslant x \leqslant \max\Big(e_2(k^c), e_2\Big(\frac{1}{E(k)^3}\Big)\Big) \tag{23}$$

if only k is "good" in the sense of (17). Omitting further results I mention only the theorem from [49] according to which, for $T > c$, there is a subinterval in $[\log_3 T, T]$ of the form

$$(U_2 e^{-\log^{9/10} U_2} \leqslant)\, U_1 < U_2 \tag{24}$$

so that

$$\sum_{\substack{U_1 \leqslant p \leqslant U_2 \\ p\equiv 1 \bmod 4}} \log p - \sum_{\substack{U_1 \leqslant p \leqslant U_2 \\ p \equiv 3 \bmod 4}} \log p > \sqrt{U_2}, \tag{25}$$

(⁷) In the paper [38] this theorem is stated much more explicitely.

which owing to (20) again does *not* strengthen the belief that $L(s, 4, \chi_1)$ does not vanish for $\sigma > \frac{1}{2}$.

I should like to mention two more of Knapowski's papers dealing with analytical number theory (not chronologically). The first one is his posthumous paper [50]. It is well-known since forty years that with a suitable positive numerical c_0 the function $\prod_\chi L(s, k, \chi)$ can have at most one single zero in the segment

$$(26) \qquad 1 - \frac{c_0}{\log(k+1)} \leqslant s < 1.$$

The possibility of this zero gives a lot of trouble; it is called (if exists) a *Siegel zero*, since Siegel proved first that if it exists, then it is

$$(27) \qquad \leqslant 1 - \frac{c_1(\varepsilon)}{k^\varepsilon}$$

for arbitrarily small positive ε, where — curiously enough — $c_1(\varepsilon)$ is an *neffective* constant. Besides Siegel's proof also Estermann, S. Chowla and Linnik gave proofs for this important theorem; the paper [50] contains an ingenious short proof using the theorem (5)-(6). The second paper (see [21]) deals with Linnik's theorem (explained in the footnote ([3])), whose proof — in the shortened version of K. A. Rodosskiĭ — takes the last forty pages of the well known K. Prachar's book "Primzahlverteilung". The proof consists of two parts essentially each of about the same length. The first one is a density theorem, the second one is the elimination of a difficulty caused by the possible appearance of the Siegel root, from these two parts the proof follows quickly. I have found in 1960 that the density theorem part can be proved shortly using theorem (5)-(6), and have risked the conjecture that the same can be done with the second part. In the paper [21] Knapowski confirmed this conjecture using again ingeniously the theorem (5)-(6).

I am going to mention something on the papers [52] and [53] which will have the title "Further developments in the comparative prime number theory", VII and VIII. In all applications of the one-sided theorem condition (11) is that which makes headache. We found in the paper [45], using also an ingenious idea of G. Kreisel, that the *two-sided* theorem (5)-(6) can produce *one-sided* theorem of the type (25). However, the localisation of x_1, x_2 for which we could prove (with the notation (21)) that

$$\psi(x_1, k, 1) - \psi(x_1, k, l) > x_1^{1/2-\delta},$$
$$\psi(x_2, k, 1) - \psi(x_2, k, l) < -x_2^{1/2-\delta}$$

was rather weak, about of the type

$$(28) \qquad (a, e^{\log^2 a (\log\log a)^3}).$$

Now in paper [52] it will be shown by the *two-sided* theorem that for "good" k-moduli if for a character χ with $\chi(l) \neq 1$ the $L(s, k, \chi)$ function has a zero

$$\varrho_0 = \beta_0 + i\gamma_0, \quad \beta \geqslant \tfrac{1}{2}, \ \gamma_0 > 0,$$

then for (8)

$$a > \max\left(c, e_2(k), e_1(\gamma_0^4)\right), e_1\left(E(k)^{-60}\right)$$

there hold rather well-localised inequalities

$$\max_{a \leqslant x \leqslant a \exp\{\log^{3/4} a (\log\log a)^3\}} \{\psi(x, k, 1) - \psi(x, k, l)\} > a^{\beta_0 - E^2/8\sqrt{\log a}}, \tag{29}$$

$$\min_{a \leqslant x \leqslant a \exp\{\log^{3/4} a (\log\log a)^3\}} \{\psi(x, k, 1) - \psi(x, k, l)\} < -a^{\beta_0 - E^2/8\sqrt{\log a}}.$$

As to paper [54], it will deal with a modification of theorem in (20) for the Čebyševian case $k = 4$. The full analogy with it is marked by the presence of the $\log p$ factors. In paper [54] it will be shown that assertion (16) is fully equivalent to

$$\lim_{x \to +\infty} \sum_{p > 2} (-1)^{(p-1)/2} e^{-\log^2 p/x} = -\infty, \tag{30}$$

which — together with (24)-(25) — makes still more doubtful the truth of the assertion (16). We could do it for all "good" k-moduli.

Most of the time our mathematical connection was by correspondence. Though this was mainly mathematical, his style and even handwriting indicated from the very beginning his highly cultured personality. This impression was confirmed by the personal contact. With the assistance of the Polish Academy of Sciences he made longer visits in Hungary; we met at my short visits in Poland, e.g. I had the honor to be one of his opponents of his docent examination at the Poznań-University in 1960. But also we met each other at some Western Universities; particularly fruitful were the summer of 1963 and 1964 at Ann Arbor and Columbus Ohio. The long talks during evening strolls, whose main theme was mathematics and especially the further course of our joint work, was intermingled with discussions on music, literature and life, discussion which carefully concealed the serious behind jokes and usually ended before returning to home at a Student-Association Building with piano where he played Chopin and Liszt attracting a large audience. Car driving was one of his main hobbies; we made large excursions by car and according to my experiences he was a safe driver (apart from a single occasion). It certainly did not occur to me that this will be fatal for him.

The indications of his activity in number theory do not give a full mathematical picture of Knapowski. To illustrate his ability to react

(8) As to $E(k)$, see (18).

quickly and effectively also in other parts of mathematics, I can give some other examples. N. Wiener showed that if a trigonometrical series

$$(31) \qquad \sum_{n=-\infty}^{\infty} a_n e^{\lambda_n it}, \qquad \lambda_{n+1}-\lambda_n \to +\infty,$$

with integer increasing λ_n's is, e.g., Abel-summable in an arbitrarily small interval I to an $f(x) \in L_2(I)$, then it is Abel-summable almost everywhere to a function $f_1(x)$ belonging to $L_2(-\pi, \pi)$ (and series (31) is its Fourier series). Zygmund and Marcinkiewicz proved this in a simpler way; the former raised and proposed more than once to Erdös the problem what happens if the condition $f(x) \in L_2(I)$ is replaced by $f(x) \in L_q(I)$ with $q > 2$. Erdös and Rényi showed by probabilistic arguments (and thus without exhibiting an explicit example) that the answer for all $q > 2$ is the opposite to the case $q = 2$. They postulated to give explicit examples to this phenomenon. I gave a tricky example of the required sort (even with much bigger gaps than in (31), but for $q > 6$ only, and mentioned to Knapowski the question how one can improve my construction in order to increase this range of q. Though he did not possess any previous experience in such techniques (to the best of my knowledge) he improved it within two weeks to $q > 3$ by further non-technical ideas (see [39]).

A similar experience was told to me by Alexits in connection with their joint paper [52] in approximation theory.

By these few lines I have intended to indicate the main lines of our collaboration; it was not the task to assess his whole work or to write his whole biography. It does not say a word on him as a teacher, it does not say anything on his devotion for Poland (which I know). He became one of my best friends in the course of years and these lines want also to reflect my sorrow for the untime death of my "mathematical son" as he called himself with friendly exaggeration in one of his letters.

LIST OF SCIENTIFIC PAPERS OF STANISŁAW KNAPOWSKI

[1] *On the greatest prime factors of certain products*, Annales Polonici Mathematici 2 (1955), p. 56-63.

[2] *O pewnych kryteriach nierozkładalności wielomianów* [On some criteria of indecomposability of polynomials], Prace Matematyczne 1 (1955), p. 272-275. [Polish]

[3] *Twierdzenie z teorii grup skończonych* [A theorem from the finite group theory], Prace Matematyczne 2 (1956), p. 165-166. [Polish]

[4] *Kryterium nierozkładalności równań stopnia* $p+1$ [A criterion of indecomposability of equalities of degree p + 1], Prace Matematyczne 2 (1956), p. 170-171. [Polish]

[5] (und S. Hartman) *Bemerkungen über die Bruchteile von pa*, Annales Polonici Mathematici 3 (1957), p. 285-287.

[6] *Zur Arithmetik der Polynome*, The Journal of the London Mathematical Society 32 (1957), p. 319-321.

[7] *Über ein Problem der Gleichverteilungstheorie*, Colloquium Mathematicum 5 (1957), p. 8-10.

[8] *On prime numbers in an arithmetical progression*, AA 4 (1958)*, p. 57-70.

[9] *On the Möbius function*, AA 4 (1958), p. 209-216.

[10] *On the distribution of prime-numbers in the sequences of positive integers*, Bulletin de la Société des Amis des Sciences et des Letters de Poznań, XIV, Sér. B, (1958), p. 9-30.

[11] *On new "explicit formulas" in prime-number theory I*, AA 5 (1958), p. 1-14.

[12] *On an explicit lower estimate in prime-number theory*, The Journal of the London Mathematical Society 34 (1959), p. 437-441.

[13] *On the mean values of certain functions in prime-number theory*, AMH 10 (1959), p. 375-390.

[14] *On the zeros of certain Dirichlet polynomials*, Indagationes Mathematicae 22 (1960), p. 54-58.

[15] *On new "explicit formulas" in prime-number theory II*, AA 6 (1960), p. 23-35.

[16] *Contributions to the theory of the distribution of prime-numbers in arithematical progressions I*, AA 6 (1961), p. 415-434.

[17] *On sign-changes in the remainder-term in the prime-number formula*, The Journal of the London Mathematical Society 36 (1961), p. 451-460.

[18] *Mean-value estimations for the Möbius function I*, AA 7 (1962), p. 121-130.

[19] *On sign-changes of the difference* $\pi(x) - \mathrm{li}\, x$, AA 7 (1962), p. 107-119.

[20] (and W. Staś) *A note on a theorem of Hardy and Littlewood*, AA 7 (1962), p. 161-166.

[21] *On Linnik's theorem concerning exceptional L-zeros*, Publicationes Mathematicae 9 (1962), p. 168-178.

[22] *Contributions to the theory of the distribution of prime-numbers in arithmetical progressions II*, AA 7 (1962), p. 325-335.

* Abbreviations: AA — Acta Arithmetica, AMH — Acta Mathematica Academiae Scientiarum Hungaricae.

[23] *Mean-value estimations for the Möbius function II*, AA 7 (1962), p. 337-343.

[24] *Contributions to the theory of the distribution of prime-numbers in arithmetical progressions III*, AA 8 (1962), p. 97-105.

[25] (and P. Turán) *Comparative prime-number theory I*, AMH 13 (1962), p. 299-314.

[26] (and P. Turán) *Comparative prime-number theory II*, AMH 13 (1962), p. 315-342.

[27] (and P. Turán) *Comparative prime-number theory III*, AMH 13 (1962), p. 343-364.

[28] (and W. Staś) *Another note on Hardy-Littlewood's theorem*, AA 8 (1963), p. 205-212.

[29] *Przegląd niektórych zagadnień analitycznej teorii liczb dotyczących rozkładu liczb pierwszych* [*A survey of some problems from the analytical number theory concerning decomposition of primes*], Wiadomości Matematyczne 6 (1963), p. 115-134. [Polish]

[30] (and P. Turán) *Comparative prime-number theory IV*, AMH 14 (1963), p. 31-42.

[31] (and P. Turán) *Comparative prime-number theory V*, AMH 14 (1963), p. 43-63.

[32] (and P. Turán) *Comparative prime-number theory VI*, AMH 14 (1963), p. 65-78.

[33] (and P. Turán) *Comparative prime-number theory VII*, AMH 14 (1963), p. 241-250.

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