COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 2

CONNECTIONS PARTIALLY ADAPTED TO A METRIC ($J^4 = 1$)-STRUCTURE

BY

E. REYES (VALLADOLID), A. MONTESINOS (BURJASOT, VALENCIA) AND P. M. GADEA (MADRID)

1. Introduction. The motivation for the study of ($J^4 = 1$)-structures is twofold. First, they can be considered as a generalization of the first-class electromagnetic (1, 1) tensor field $\tilde{J}$ of Hlavatý [3] and Mishra [5]. Also, it is a structure that combines those of almost product and almost complex manifolds.

Let $\tilde{J}$ be a (1, 1) tensor field on a differentiable manifold M^n. We say that it defines a ($J^4 = 1$)-*structure* if there are everywhere non-zero C^∞-functions p and q on M^n such that

$$(\tilde{J}^2 - p^2)(\tilde{J}^2 + q^2) = 0,$$

and positive integer numbers r_1, r_2, s ($r_1 + r_2 + 2s = n$) such that

$$(x-p)^{r_1}(x+p)^{r_2}(x^2+q^2)^s$$

is the characteristic polynomial of $\tilde{J}$.

From the first point of view it is natural to impose the existence of a pseudo-Riemannian metric g on M such that

$$g(\tilde{J}X, Y) + g(X, \tilde{J}Y) = 0;$$

then, the (0, 2) tensor field F defined by $F(X, Y) = g(\tilde{J}X, Y)$ is a non-degenerate 2-form that in the case $n = 4$ represents the electromagnetic field. Due to this, we say that g is an "aem" (adapted in the electromagnetic sense metric). The conditions for the existence of an aem for a given $\tilde{J}$ are studied in [2].

From the other perspective, we consider a Riemannian metric g which is Hermitian upon the almost complex subbundle and makes pairwise orthogonal the three subbundles determined by $\tilde{J}$ in TM. This will be called an adapted Riemannian metric or, briefly, "arm".

In [2] it is proved that the G-structure P defined by ($\tilde{J}$, g) (in both cases) can also be obtained replacing $\tilde{J}$ by another (1, 1) tensor field J that satisfies $J^4 = 1$ (this motivates the name ($J^4 = 1$)-structures). It is given by

$$J = \frac{p^3+q^3}{pq(p^2+q^2)}\tilde{J} + \frac{q-p}{pq(p^2+q^2)}\tilde{J}^3.$$

The new tensor field J is 0-deformable and can be looked at as a tool for the study of the G-structure P. Thus, in [2] the integrability of P in terms of J is studied. Also, there is no linear connection parallelizing $\tilde{J}$ unless p and q were constants, and so in [7] we give the family of connections that parallelize both J and g.

In this paper we study some special connections that are only partially adapted to the G-structure (J, g) but have instead other nice geometrical properties. They generalize in some sense the well-known special connections in almost product and almost Hermitian structures.

2. The connection D. In the following we assume that J is a (1, 1) tensor field on M^n with the characteristic polynomial

$$(x-1)^{r_1}(x+1)^{r_2}(x^2+1)^s, \qquad r_1+r_2+2s = n,$$

and such that $J^4 = 1$. Let

$$l_1 = \tfrac{1}{4}(1+J)(1+J^2), \qquad l_2 = \tfrac{1}{4}(1-J)(1+J^2),$$
$$l = \tfrac{1}{2}(1+J^2), \qquad l_3 = \tfrac{1}{2}(1-J^2)$$

be the projectors that define the direct sums of vector bundles

$$TM^n = L_1 \oplus L_2 \oplus L_3, \quad L = L_1 \oplus L_2.$$

Then

$$JX = X, \quad JX = -X, \quad J^2X = X, \quad J^2X = -X$$

if, respectively,

$$X \in L_1, \quad X \in L_2, \quad X \in L, \quad X \in L_3.$$

We consider first the case of an arm. An arm is a Riemannian metric g on M^n such that

$$g(JX, JY) = g(X, Y) \quad \text{for } X, Y \in T_xM^n,\ x \in M^n.$$

Then L_1, L_2, L_3 are pairwise orthogonal.

There always exists an arm in a ($J^4 = 1$)-manifold, as is obvious. We denote by D the Levi-Cività connection of the arm g. Then we have obviously

PROPOSITION 2.1. *If D induces connections in any two of the subbundles L_1, L_2, L_3, then it induces a connection in the third one, and moreover the three subbundles are integrable.*

The property of inducing connections in L_1, L_2 and L_3 can be characterized according to the

PROPOSITION 2.2. *A linear connection ∇ in TM^n induces connections in L_1, L_2, L_3 iff*

$$\nabla_X l_3 = \nabla_X l = l\nabla_X J = 0.$$

Proof. If $\nabla_X l = \nabla_X l_3 = 0$, we have $\nabla_X lY \in L$ and $\nabla_X l_3 Y \in L_3$. If also $l\nabla_X J = 0$, then $\nabla_X lJ = 0$; thus $\nabla_X l_1 = \nabla_X l_2 = 0$, and so $\nabla_X l_i Y \in L_i$, $i = 1, 2$.

Conversely, if $\nabla_X l_i Y \in L_i$, $i = 1, 2$, then $\nabla_X lY \in L$ for all Y, and so

$$l_3 \nabla_X lY = l_3(\nabla_X l) Y + l_3 l\nabla_X Y = l_3(\nabla_X l) Y = \tfrac{1}{2} l_3(\nabla_X J^2) Y = 0.$$

Thus $l_3 \nabla_X J^2 = 0$. Analogously, $l\nabla_X J^2 = 0$. Summing up, we have $\nabla_X J^2 = 0$, and so $\nabla_X l = \nabla_X l_3 = 0$. But, since

$$l_2 \nabla_X l_1 = l_1 \nabla_X l_2 = 0$$

and

$$l_1 = \tfrac{1}{2}(1+J)l, \quad l_2 = \tfrac{1}{2}(1-J)l, \quad \nabla l = 0,$$

we have

$$l_1 \nabla_X(Jl) = l_1 l\nabla_X J = 0 \quad \text{and} \quad l_2 l\nabla_X J = 0.$$

Summing up we have

$$l^2 \nabla_X J = l\nabla_X J = 0.$$

We note that we do not conclude $l_3 \nabla_X J = 0$. That is, the earlier conditions do not guarantee $\nabla J = 0$.

3. The connection ∇. We suppose now that g is an arm. Then we have

PROPOSITION 3.1. *There is a unique linear connection ∇ on (M^n, J, g) such that*

(a) *∇ induces connections ∇^1, ∇^2, ∇^3 in L_1, L_2, L_3, respectively; that is,*

$$\nabla_X l_i Y \in L_i, \quad i = 1, 2, 3.$$

(b) *The induced connections are Euclidean along the tangent curves to L_1, L_2, L_3, respectively; that is,*

$$(\nabla_{l_i X} g)(l_i Y, l_i Z) = 0, \quad i = 1, 2, 3.$$

(c) *$T_i(X, l_i Y) = 0$, where $T_i = l_i T$, $i = 1, 2, 3$.*

Proof. If we make the decomposition

$$\nabla_X Y = \nabla_{l_1 X} l_1 Y + \nabla_{l_2 X} l_1 Y + \nabla_{l_3 X} l_1 Y + \nabla_{l_1 X} l_2 Y + \nabla_{l_2 X} l_2 Y + \nabla_{l_3 X} l_2 Y$$
$$+ \nabla_{l_1 X} l_3 Y + \nabla_{l_2 X} l_3 Y + \nabla_{l_3 X} l_3 Y,$$

the condition (c) determines the 2nd, 3rd, 4th, 6th, 7th, 8th terms in the right-hand side, and the condition (b) determines the 1st, 5th and 9th terms.

We have also

PROPOSITION 3.2. *The connection* ∇ *is torsionless iff* L_1, L_2, L, L_3, $L_1 \oplus L_3$, $L_2 \oplus L_3$ *are integrable.*

Proof. It is clear that

$$T_i(l_j X, l_j Y) = -l_i[l_j X, l_j Y], \quad i \neq j,$$

and

$$T_i(l_j X, l_k Y) = -l_i[l_j X, l_k Y], \quad k \neq i \neq j \neq k, \; i, j, k = 1, 2, 3.$$

The proposition is immediate from this.

We can now see the relation between D and ∇.

PROPOSITION 3.3. ∇ *is identical to* D *iff* D *induces connections in* L_1, L_2, L_3.

The proof is immediate from the properties of ∇.

The connection ∇ has the following property:

PROPOSITION 3.4. *A geodesic of* ∇ *is tangent at every point to one of the subbundles* L_1, L_2 *or* L_3 *iff it is tangent to it at one point.*

This proposition is a direct consequence of the following general result:

LEMMA 3.1. *Let* $\mathscr{D}$ *be a subbundle of* TM^n *and* ∇ *a linear connection in* M. *Then every geodesic with initial condition in* $\mathscr{D}$ *maintains its tangent in* $\mathscr{D}$ *iff* $\nabla_X X \in \mathscr{D}$ *for* $X \in \mathscr{D}$.

Proof. Let ∇ be a linear connection on M^n. Let $\{e_i\}$, $i = 1, \ldots, n$, be a local frame of vector fields on $U \subset M^n$, and let $\{\bar{\theta}^i\}$ be the dual coframe. The 1-forms $\bar{\theta}^i$ can be considered as functions in TU, and so we have a trivialization

$$\pi^{-1}(U) \to U \times \mathbf{R}^n,$$

$$X_x \in T_x U \rightsquigarrow (x, \bar{\theta}^i(X_x)).$$

Thus, a vector field on TU can be locally represented as

$$\tilde{X} = a^i e_i + b^i \frac{\partial}{\partial \theta^i},$$

where a^i, b^i $(i = 1, \ldots, n)$ are functions and θ^i are the canonical coordinates in $\mathbf{R}^n$.

If we have $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ in U, then the spray of ∇ is the vector field $\tilde{G}$ in TM^n that we can represent in TU as

$$\tilde{G} = \theta^i e_i - \Gamma_{jk}^i \theta^j \theta^k \frac{\partial}{\partial \theta^i}.$$

$\tilde{G}$, as is well known, is the vector field in TM^n whose integral curves are projected in M^n giving the geodesics of ∇.

Now, let $\mathscr{D}$ be a q-dimensional subbundle of TM^n. We can consider $\mathscr{D}$ as a regular submanifold of TM^n. Thus, every geodesic with initial condition in $\mathscr{D}$ maintains its tangent in $\mathscr{D}$ iff $\tilde{G}|_{\mathscr{D}}$ is tangent to $\mathscr{D}$.

Then, let $\{e_i\}$ be such that $\{e_a\}$, $a = 1, \ldots, q$, generates $\mathscr{D}$ in U, and consider $x \in U$, $X_x \in \mathscr{D}$ and $\tilde{X} \in T_{X_x} TU$. Then $\tilde{X}$ is tangent to $\mathscr{D}$ iff there is a curve

$$\sigma(t) = \sigma^i(t)\, e_{i(\pi\cdot\sigma)(t)}$$

such that $\sigma(0) = X_x$, $\sigma(t) \in \mathscr{D}$ in a neighbourhood of $0 \in \boldsymbol{R}$, and $\dot{\sigma}(0) = \tilde{X}$. In order that $\sigma(t) \in \mathscr{D}$ we put

$$\sigma(t) = \sigma^a(t)\, e_{a(\pi\cdot\sigma)(t)}.$$

Thus

$$\dot{\sigma} = \dot{\overline{\pi\cdot\sigma}} + \dot{\sigma}^a\left(\frac{\partial}{\partial\theta^a}\cdot\sigma\right).$$

If

$$\tilde{X} = a^i e_{ix} + b^i \left.\frac{\partial}{\partial\theta^i}\right|_{X_x},$$

we have

$$a^i e_{ix} = \dot{\overline{(\pi\cdot\sigma)}}(0), \qquad b^u = 0,\ u = q+1, \ldots, n, \qquad b^a = \dot{\sigma}^a(0).$$

But at X_x we have

$$\tilde{G}_{X_x} = \theta^i(X_x)\, e_{ix} - (\Gamma^i_{jk})_x\, \theta^j(X_x)\, \theta^k(X_x) \left.\frac{\partial}{\partial\theta^i}\right|_{X_x},$$

and since $X_x \in \mathscr{D}$, we deduce

$$X_x = X^a_x e_{ax},$$

that is

$$\tilde{G}_{X_x} = X^a_x e_{ax} - (\Gamma^i_{ab})_x X^a_x X^b_x \left.\frac{\partial}{\partial\theta^i}\right|_{X_x}.$$

Thus, in order that $\tilde{G}_{X_x}$ should be tangent to $\mathscr{D}$ it is necessary that

$$\Gamma^i_{ab} + \Gamma^i_{ba} = 0, \qquad i > q.$$

The condition is clearly sufficient, since $X_x \in \mathscr{D}$. More intrinsically, we can give the condition as

$$\nabla_X X \in \mathscr{D} \quad \text{if } X \in \mathscr{D}.$$

Indeed, if $X \in \mathscr{D}$, then $X = X^a e_a$, and so

$$\nabla_X X = X^a \nabla_{e_a}(X^b e_b) = X^a e_a(X^b) e_b + X^a X^b \nabla_{e_a} e_b$$
$$= X(X^b) e_b + \tfrac{1}{2}(\Gamma^i_{ab} + \Gamma^i_{ba}) X^a X^b e_i,$$

and it is in $\mathscr{D}$ for every $X \in \mathscr{D}$ iff $\Gamma^i_{ab} + \Gamma^i_{ba} = 0$, since $X(X^b) e_b \in \mathscr{D}$.

4. The connection $\tilde{\nabla}$. We consider again an arm g on M^n, but now we consider the almost complex operator induced by J on L_3.

PROPOSITION 4.1. *There is a unique linear connection $\tilde{\nabla}$ on (M^n, J, g) such that*

(a) $\tilde{\nabla}_X l_i Y \in L_i$, $i = 1, 2, 3$;
(b) $(\tilde{\nabla}_{l_i X} g)(l_i Y, l_i Z) = 0$, $i = 1, 2, 3$;
(c) $T_i(X, l_i Y) = 0$, $i = 1, 2$;
(d) $T_3(JX, l_3 Y) = T_3(X, Jl_3 Y)$;
(e) $(\tilde{\nabla}_{l_3 X} J) l_3 Y = 0$.

That is, as in the case of ∇, we require that by restriction we get linear connections in the structural subbundles, and also that the parallel transport along a tangent curve to each of the subbundles preserves the scalar product of vectors of such a subbundle; in particular, that the length of such a vector is preserved. The conditions on the torsion are the same as before in the case $i = 1, 2$, but in L_3 we consider the usual conditions of the almost Hermitian case.

Proof. If we make a decomposition similar to that given in the proof of Proposition 3.1, we obtain 9 terms; the terms 1–6 are determined as in Proposition 3.1; the terms 7 and 8 by means of the condition (d), and the last term in the usual way (see [9] and also Theorem 5.1 below).

Now, we see the relation between ∇ and $\tilde{\nabla}$.

We define the 2-form F as

$$F(X, Y) = g(l_3 X, Jl_3 Y),$$

and put

$$N(X, Y) = N_{Jl_3}(l_3 X, l_3 Y).$$

Then as usual we have (see [4], p. 148)

(4.1)
$$4g((\nabla_{l_3 X} J) l_3 Y, l_3 Z) = 2dF(Jl_3 X, l_3 Z, Jl_3 Y) + 2dF(Jl_3 X, Jl_3 Z, l_3 Y)$$
$$+ g(N(Jl_3 Y, l_3 X), l_3 Z) + g(N(l_3 X, Jl_3 Z), l_3 Y),$$

and if we define $l_3 dF$ as

$$(l_3 dF)(X, Y, Z) = dF(l_3 X, l_3 Y, l_3 Z),$$

we have

PROPOSITION 4.2. (i) $\nabla = \tilde{\nabla}$ *iff* $l_3 N = 0$ *and* $l_3\, dF = 0$.

(ii) L_3 *is integrable and its leaves are Kaehlerian iff* $N = 0$ *and* $l_3\, dF = 0$.

(iii) $D = \tilde{\nabla}$ *iff* D *induces connections in* L_1, L_2, L_3 *and* $\nabla = \tilde{\nabla}$.

Proof. (i) Suppose $\nabla = \tilde{\nabla}$. Then

$$dF(l_3 X, l_3 Y, l_3 Z) = \mathrm{cycl}(\nabla_{l_3 X} F)(l_3 Y, l_3 Z)$$
$$= \mathrm{cycl}\,\{l_3 X(g(l_3 Y, Jl_3 Z)) - g(\nabla_{l_3 X} l_3 Y, Jl_3 Z) - g(l_3 Y, \nabla_{l_3 X} Jl_3 Z)\} = 0$$

because of (b) in Proposition 3.1 or 4.1. On the other hand, as is easily verified,

$$N(X, Y) = -l[l_3 X, l_3 Y],$$

and thus

$$l_3 N(X, Y) = 0.$$

Conversely, if $l_3 N = 0$ and $l_3 dF = 0$, we deduce from (4.1) that

$$(\nabla_{l_3 X} J) l_3 Y = 0,$$

and since $l_3 T(JX, l_3 Y) = 0 = l_3 T(X, Jl_3 Y)$, we have $\nabla = \tilde{\nabla}$.

(ii) If $N = 0$ and $l_3\, dF = 0$, then $\nabla = \tilde{\nabla}$, whence

$$N(X, Y) = -l[l_3 X, l_3 Y] = 0,$$

and thus L_3 is integrable. Also, since $(\nabla_{l_3 X} J) l_3 Y = 0$, the leaves of L_3 are Kaehlerian. Conversely, if L_3 is integrable and its leaves are Kaehlerian, we have

$$T(l_3 X, l_3 Y) = l_3 T(l_3 X, l_3 Y) = 0,$$

and so $\nabla_{l_3 X} l_3 Y = D_{l_3 X} l_3 Y$. Since

$$(D_{l_3 X} J) l_3 Y = D_{l_3 X} J l_3 Y - J D_{l_3 X}\, l_3 Y = (\nabla_{l_3 X} J) l_3 Y = 0,$$

we obtain $\nabla = \tilde{\nabla}$, and thus

$$l_3\, dF = 0 \quad \text{and} \quad N(X, Y) = -l[l_3 X, l_3 Y] = 0.$$

(iii) If $D = \tilde{\nabla}$, then $T_3 = 0$, and thus $\tilde{\nabla}$ satisfies the conditions of $\tilde{\nabla}$, whence $\nabla = \tilde{\nabla} = D$. The rest follows by direct application of Proposition 3.3.

5. The aem case: the connection ∇. We now suppose that g is an aem, that is a pseudo-Riemannian metric such that

$$g(JX, Y) = -g(X, JY)$$

(see [2] for the conditions of the existence of such a metric).

We require, as in the previous cases of ∇ and $\tilde{\nabla}$, that ∇ induces connections in the subbundles L_1, L_2 and L_3, and so ∇ is the sum of three connections ∇^1, ∇^2 and ∇^3 or, if we consider only $TM^n = L \oplus L_3$, the sum of two connections ∇^L and ∇^3, that we shall study separately. We adopt from now on the following notation:

$A, B, C, \ldots$ are vectors and vector fields of L;

$A_1, B_1, C_1, \ldots$ are vectors and vector fields of L_1;

$A_2, B_2, C_2, \ldots$ are vectors and vector fields of L_2;

$X, Y, Z, \ldots$ are vectors and vector fields of L_3;

and we recall that $L \perp L_3$, $L_1 \perp L_1$ and $L_2 \perp L_2$.

We have a canonical partial connection ∇^3_L given by

$$\nabla^3_A X = l_3 [A, X],$$

which in the case where L were a foliation coincides with Bott's partial connection [1].

As for the existence and uniqueness of ∇^3_3 we can apply Vaisman's construction (see Proposition 4.1), since both for aem and arm we have

$$g(JX, Y) + g(X, JY) = 0.$$

But in order to obtain an explicit expression for ∇^3_3 we give another proof in the following

THEOREM 5.1. *There exists a unique partial connection ∇^3_3 in the subbundle L_3 which is Hermitian in the sense that*

(a) $(\nabla^3_X g)(Y, Z) = 0$;

(b) $\nabla^3_X JY = J\nabla^3_X Y$;

(c) $T_3(JX, Y) = T_3(X, JY)$, *where* $T_3(X, Y) = \nabla^3_X Y - \nabla^3_Y X - l_3[X, Y]$.

Proof. We observe first that $K = J|_{L_3}$ is a section of End L_3 such that $K^2 = -1$ and that $g|_{L_3}$ defines a metric such that

$$g(KX, Y) + g(X, KY) = 0.$$

Let ∇ be a partial connection in L_3 (that is, ∇ is defined by means of $\nabla_X Y$ with the usual conditions) and let H be a (1, 2) tensor field in L_3, that is, $H(X, Y) \in L_3$ and it is defined only for vector fields of L_3. Then $\nabla_X Y + H(X, Y)$ defines a new L_3-partial connection in L_3. Following Obata [6] we define the operators ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ as follows:

$$(\phi\nabla)_X Y = \nabla_X Y - \tfrac{1}{2} K(\nabla_X K) Y,$$

$$(\tilde{\phi} H)(X, Y) = \tfrac{1}{2} H(X, Y) - \tfrac{1}{2} K H(X, KY),$$

$$g((\psi\nabla)_X Y, Z) = g(\nabla_X Y, Z) + \tfrac{1}{2}(\nabla_X g)(Y, Z),$$

$$g((\tilde{\psi} H)(X, Y), Z) = \tfrac{1}{2} g(H(X, Y), Z) - \tfrac{1}{2} g(Y, H(X, Z)).$$

Then $\phi\nabla$ and $\psi\nabla$ are L_3-partial connections in L_3 and we have:

(i) If $\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y)$, then

$$\phi\tilde{\nabla} = \phi\nabla + \tilde{\phi} H \quad \text{and} \quad \psi\tilde{\nabla} = \psi\nabla + \tilde{\psi} H.$$

(ii) $\phi\psi = \psi\phi$.

(iii) An L_3-partial connection $\tilde{\nabla}$ in L_3 satisfies $\tilde{\nabla}g = 0$ and $\tilde{\nabla}K = 0$ iff there exists another connection ∇ such that $\tilde{\nabla} = \phi\psi\nabla$.

(iv) Let ∇ be an arbitrary (but fixed) L_3-partial connection in L_3. Then the following expression, for any H, gives all (and only) the L_3-partial connections in L_3 such that $\tilde{\nabla}g = 0$ and $\tilde{\nabla}J = 0$:

$$\tilde{\nabla} = \psi\phi\nabla + \tilde{\psi}\tilde{\phi}H.$$

On the other hand, since the torsion of an L_3-partial connection ∇ in L_3 is

$$T_3(X, Y) = \nabla_X Y - \nabla_Y X - l_3[X, Y]$$

we see, if

$$\tilde{\nabla} = \nabla + \tilde{\psi}\tilde{\phi}H,$$

that

$$\tilde{T}_3(X, Y) = T_3(X, Y) + (\tilde{\psi}\tilde{\phi}H)(X, Y) - (\tilde{\psi}\tilde{\phi}H)(Y, X),$$

and thus

$$\begin{aligned}(5.1)\qquad \tilde{T}_3(KX, Y) - \tilde{T}_3(X, KY) = {} & T_3(KX, Y) - T_3(X, KY) + (\tilde{\psi}\tilde{\phi}H)(KX, Y) \\ & -(\tilde{\psi}\tilde{\phi}H)(Y, KX) - (\tilde{\psi}\tilde{\phi}H)(X, KY) + (\tilde{\psi}\tilde{\phi}H)(KY, X).\end{aligned}$$

Let D be the Levi–Cività connection in TM^n corresponding to g and put

$$\hat{\nabla}_X Y = l_3 D_X Y.$$

Then $\hat{\nabla}$ is the L_3-partial connection in L_3 verifying $\hat{\nabla}g = 0$ and $\hat{T}_3(X, Y) = 0$, as is easily proved.

Now, we consider

$$\nabla = \psi\phi\hat{\nabla} = \phi\hat{\nabla}$$

(since $\hat{\nabla}g = 0$), and thus

$$\nabla_X Y = \hat{\nabla}_X Y - \tfrac{1}{2}K(\hat{\nabla}_X K)Y.$$

Then, since $\hat{T}_3(X, Y) = 0$, we obtain

$$\begin{aligned}(5.2)\qquad T_3(X, Y) &= \nabla_X Y - \nabla_Y X - l_3[X, Y] \\ &= \hat{\nabla}_X Y - \tfrac{1}{2}K(\hat{\nabla}_X K)Y - \hat{\nabla}_Y X + \tfrac{1}{2}K(\hat{\nabla}_Y K)X - l_3[X, Y] \\ &= \tfrac{1}{2}K(\hat{\nabla}_Y K)X - \tfrac{1}{2}K(\hat{\nabla}_X K)Y.\end{aligned}$$

We then have

$$\begin{aligned}(5.3)\qquad 4g((\tilde{\psi}\tilde{\phi}T_3)(X, Y), Z) &= 2g((\tilde{\phi}T_3)(X, Y), Z) - 2g(Y, (\tilde{\phi}T_3)(X, Z)) \\ &= g(T_3(X, Y), Z) - g(Y, T_3(X, Z)) \\ &\quad - g(KT_3(X, KY), Z) + g(Y, KT_3(X, KZ))\end{aligned}$$

and, as is proved by a computation (see the Appendix),

$$(5.4)\quad 4g((\tilde\psi\tilde\phi T_3)(KX, Y), Z)-4g((\tilde\psi\tilde\phi T_3)(Y, KX), Z)$$
$$-4g((\tilde\psi\tilde\phi T_3)(X, KY), Z)+4g((\tilde\psi\tilde\phi T_3)(KY, X), Z)$$
$$= 2g(T_3(KX, Y)-T_3(X, KY), Z).$$

Compare now the right-hand side members in (5.4) and (5.1). Consequently, if we put

$$H(X, Y) = -2T_3(X, Y),$$

for $\tilde\nabla = \nabla+\tilde\psi\tilde\phi H$ we have

$$(5.5)\qquad \tilde T_3(KX, Y)-\tilde T_3(X, KY) = 0,$$

and thus the required connection exists, since

$$\tilde\nabla = \nabla+\tilde\psi\tilde\phi H = \psi\phi\hat\nabla+\tilde\psi\tilde\phi H,$$

and it suffices to consider the earlier property (iv), since $\hat\nabla$ is L_3-partial, and we have (5.5).

Explicitly, we see, using (5.2) and (5.3), that

$$\tilde\nabla_X Y = \nabla_X Y+(\tilde\psi\tilde\phi T_3)(X, Y)$$

can be expressed, in terms of the Levi–Civitá connection, as

$$(5.6)\quad g(\tilde\nabla_X Y, Z) = g\big(l_3 D_X Y-\tfrac{1}{2}Jl_3(D_X J)Y+\tfrac{1}{4}Jl_3(D_Y J)X+\tfrac{1}{4}l_3(D_{JY}J)X, Z\big)$$
$$-\tfrac{1}{4}g\big(Y, Jl_3(D_Z J)X+l_3(D_{JZ}J)X\big).$$

As for the uniqueness, suppose

$$\nabla g = 0, \qquad \nabla K = 0,$$

$T_3(KX, Y) = T_3(X, KY)$. If H is any (1, 2) tensor field, then for $\tilde\nabla = \psi\phi(\nabla+H)$ we have $\tilde\nabla g = 0$, $\tilde\nabla K = 0$ and

$$\begin{aligned}\tilde T_3(X, Y) &= \tilde\nabla_X Y-\tilde\nabla_Y X-l_3[X, Y]\\ &= \nabla_X Y-\nabla_Y X-l_3[X, Y]+(\tilde\phi\tilde\psi H)(X, Y)-(\tilde\phi\tilde\psi H)(Y, X)\\ &= T_3(X, Y)+(\tilde\phi\tilde\psi H)(X, Y)-(\tilde\phi\tilde\psi H)(Y, X).\end{aligned}$$

Thus $\tilde T_3(KX, Y) = \tilde T_3(X, KY)$ iff

$$(\tilde\phi\tilde\psi H)(KX, Y)-(\tilde\phi\tilde\psi H)(Y, KX)-(\tilde\phi\tilde\psi H)(X, KY)+(\tilde\phi\tilde\psi H)(KY, X) = 0.$$

If we write

$$S(X, Y, Z) = g((\tilde\phi\tilde\psi H)(X, Y), Z)$$

and put KX instead of X, we obtain the necessary and sufficient condition

$$(5.7)\quad -S(X, Y, Z)+S(Y, X, Z)-S(KX, KY, Z)+S(KY, KX, Z) = 0.$$

On the other hand, it is obvious that

$$S(X, Y, Z) = -S(X, Z, Y) \tag{5.8}$$

and we have

$$S(X, KY, Z) = \tfrac{1}{4}g(H(X, KY), Z) - \tfrac{1}{4}g(H(X, Z), KY) - \tfrac{1}{4}g(H(X, Y), KZ) + \tfrac{1}{4}g(H(X, KZ), Y)$$

and

$$S(X, Y, KZ) = \tfrac{1}{4}g(H(X, Y), KZ) - \tfrac{1}{4}g(H(X, KZ), Y) - \tfrac{1}{4}g(H(X, KY), Z) + \tfrac{1}{4}g(H(X, Z), KY),$$

that is,

$$S(X, KY, Z) = -S(X, Y, KZ). \tag{5.9}$$

If we now consider the cyclic permutation of (5.7) and apply (5.8) and (5.9), we obtain

$$\begin{aligned} 0 &= \operatorname{cycl}\{S(X, Y, Z) - S(Y, X, Z) + S(KX, KY, Z) - S(KY, KX, Z)\} \\ &= 2\{S(Y, Z, X) + S(X, Y, Z) + S(Z, X, Y)\}. \end{aligned}$$

That is, if $\tilde{T}_3(KX, Y) = \tilde{T}_3(X, KY)$, we have (5.8), (5.9) and

$$\operatorname{cycl} S(X, Y, Z) = 0. \tag{5.10}$$

But then, applying (5.10) in (5.7) we have

$$\begin{aligned} 0 &= -S(X, Y, Z) - S(Y, Z, X) - S(KX, KY, Z) - S(KY, Z, KX) \\ &= S(Z, X, Y) + S(Z, KX, KY) = 2S(Z, X, Y), \end{aligned}$$

whence $S = 0$, and so $\tilde{\nabla} = \nabla$.

We now study ∇^1 and ∇^2. According to the previous considerations we decompose ∇^i, $i = 1, 2$, into two partial connections ∇^i_L and ∇^i_3, and we choose for ∇^i_3 the canonical connection, that is,

$$(\nabla^i_3)_X A_i = l_i[X, A_i], \quad i = 1, 2.$$

Thus, it rests only to compute the ∇^i_j, $i, j = 1, 2$. Since $(g|_L, J|_L)$ is not a Riemannian almost product pair, because $L_1 \perp L_1$, and $L_2 \perp L_2$, the Vaisman connection [8] cannot be used as a guide in our case.

Furthermore, since $g|_{L_1} = g|_{L_2} = 0$, we cannot work separately with L_1 and L_2, as we do with L and L_3. Thus we have

THEOREM 5.2. *There exists a unique L-partial connection ∇ in L such that:*

(a) $Ag(B, C) = g(\nabla_A B, C) + g(B, \nabla_A C)$;

(b) $\nabla_A JB = J\nabla_A B$, *and thus ∇ induces partial connections ∇^i_j, $i, j = 1, 2$, by restriction;*

(c) *the partial connections* ∇_2^1 *and* ∇_1^2 *are the canonical ones, that is,*

$$\nabla_{A_1} B_2 = l_2 [A_1, B_2] \quad \textit{and} \quad \nabla_{A_2} B_1 = l_1 [A_2, B_1].$$

Proof. From condition (a), if such a connection exists, we have

$$(5.11) \quad g(\nabla_A B + \nabla_B A, C) = Ag(B, C) + Bg(C, A) - Cg(A, B) \\ + g(\nabla_C A - \nabla_A C, B) + g(\nabla_C B - \nabla_B C, A).$$

But, applying now (b) and (c) we deduce

$$(5.12) \quad \nabla_A B - \nabla_B A - l[A, B] = T_1(A_1, B_1) - T_2(A_2, B_2) \\ - l_1 [A_2, B_2] - l_2 [A_1, B_1],$$

where

$$T_i(A_i, B_i) = \nabla_{A_i} B_i - \nabla_{B_i} A_i - l_i [A_i, B_i], \quad i = 1, 2.$$

Consequently, by substitution we have

$$\nabla_A B + \nabla_B A = 2\nabla_A B - l[A, B] + l_1 [A_2, B_2] + l_2 [A_1, B_1] \\ - T_1(A_1, B_1) - T_2(A_2, B_2).$$

Thus, from (5.11) and (5.12) we obtain

$$(5.13) \quad 2g(\nabla_A B, C) = Ag(B, C) + Bg(C, A) - Cg(A, B) \\ + g\big(l[A, B] - l_1 [A_2, B_2] - l_2 [A_1, B_1] \\ + T_1(A_1, B_1) + T_2(A_2, B_2), C\big) \\ + g\big(l[C, A] - l_1 [C_2, A_2] - l_2 [C_1, A_1] + T_1(C_1, A_1) \\ + T_2(C_2, A_2), B\big) \\ + g\big(l[C, B] - l_1 [C_2, B_2] - l_2 [C_1, B_1] + T_1(C_1, B_1) \\ + T_2(C_2, B_2), A\big).$$

From $L_1 \perp L_1$ and $L_2 \perp L_2$ we deduce that $\nabla_A JB = J\nabla_A B$ iff

$$2g(\nabla_A B_1, C_1) = 2g(\nabla_A B_2, C_2) = 0.$$

Thus, if we put $B = B_1$ and $C = C_1$ in (5.13), we obtain

$$(5.14) \quad 0 = B_1 g(C_1, A_2) - C_1 g(A_2, B_1) + g(l_2 [A_2, B_1], C_1) \\ + g\big(l_2 [C_1, A_2], B_1\big) + g\big(l_1 [C_1, B_1] + T_1(C_1, B_1), A_2\big).$$

And, since C_1, A_2 and B_1 are arbitrary, (5.10) determines completely T_1.

Indeed,

$$0 = (\mathscr{L}_{B_1} g)(C_1, A_2) - (\mathscr{L}_{C_1} g)(A_2, B_1) + 2g(l_1[B_1, C_1], A_2) + g(T_1(C_1, B_1), A_2),$$

whence

$$T_1(C_1, B_1) = 2l_1[C_1, B_1] + g^{-1}\big((\mathscr{L}_{C_1} g)(B_1, l\cdot) - (\mathscr{L}_{B_1} g)(C_1, l\cdot), \cdot\big)$$

and an analog for T_2. Thus, according to (5.13), if such a connection exists, it is unique.

Finally, the connection given by (5.13) satisfies, by construction, (a) and (b), and from the expressions of T_i condition (c) is easily deduced.

If we consider Theorems 5.1 and 5.2 simultaneously, we can give

THEOREM 5.3. *Let M^n be a (J, g)-manifold, where g is an aem. Then there exists on M^n a unique linear connection ∇ such that:*

(a) *$\nabla l = \nabla l_3 = l\nabla J = 0$, and thus ∇ is the sum of three connections ∇^1, ∇^2, ∇^3 in the vector bundles L_1, L_2, L_3, respectively, given by restriction.*

(b) *The L_3-partial connection defined from ∇^3 is Hermitian in the sense of Theorem 5.1, and the L-partial connection defined from ∇^L is adapted to J and g.*

(c) *The partial connections ∇^3_L, ∇^1_3, ∇^2_3, ∇^1_2, ∇^2_1 are the canonical ones.*

We note that

$$\nabla_{X_3} A = \nabla_{X_3} A_1 + \nabla_{X_3} A_2 = l_1[X_3, A_1] + l_2[X_3, A_2],$$

which in general is not $l[X_3, A]$. That is, the connection ∇^L_3 is not the canonical one.

We note also that it is easy to give the developed expression of ∇, and the different aspect of the L-part and the L_3-part emphasizes the fact that we do not have a Riemannian almost product structure operator on L.

APPENDIX

Verification of (5.4). The left-hand side member is

$$\begin{aligned} g(T_3(KX, Y), Z) &- g(Y, T_3(KX, Z)) - g(KT_3(KX, KY), Z) \\ &+ g(Y, KT(KX, KZ)) - g(T_3(Y, KX), Z) + g(KX, T_3(Y, Z)) \\ &+ g(KT_3(Y, K^2X), Z) - g(KX, KT_3(Y, KZ)) - g(T_3(X, KY), Z) \\ &+ g(KY, T_3(X, Z)) + g(KT(X, K^2Y), Z) - g(KY, KT_3(X, KZ)) \\ &+ g(T_3(KY, X), Z) - g(X, T_3(KY, Z)) - g(KT_3(KY, KX), Z) \\ &+ g(X, KT_3(KY, KZ)),\end{aligned}$$

where the 7th and 11th, 3rd and 15th terms cancel. Moreover, from (5.2), the 2nd, 4th, 10th and 12th terms are, respectively,

$$-g(Y, T_3(KX, Z)) = -g(Y, \tfrac{1}{2}K(\hat{\nabla}_Z K)KX - \tfrac{1}{2}K(\hat{\nabla}_{KX}K)Z);$$

$$\begin{aligned} g(Y, KT_3(KX, KZ)) &= g\big(Y, K\big(\tfrac{1}{2}K(\hat{\nabla}_{KZ}K)KX - \tfrac{1}{2}K(\hat{\nabla}_{KX}K)KZ\big)\big) \\ &= g(Y, -\tfrac{1}{2}(\hat{\nabla}_{KZ}K)KX_3 + \tfrac{1}{2}(\hat{\nabla}_{KX}K)KZ); \end{aligned}$$

$$\begin{aligned} -g(Y, KT_3(X, Z)) &= -g\big(Y, K\big(\tfrac{1}{2}K(\hat{\nabla}_Z K)X - \tfrac{1}{2}K(\hat{\nabla}_X K)Z\big)\big) \\ &= -g(Y, -\tfrac{1}{2}(\hat{\nabla}_Z K)X + \tfrac{1}{2}(\hat{\nabla}_X K)Z); \end{aligned}$$

and

$$-g(Y, T_3(X, KZ)) = g(Y, -\tfrac{1}{2}K(\hat{\nabla}_{KZ}K)X + \tfrac{1}{2}K(\hat{\nabla}_X K)KZ).$$

Thus, the sum of these terms is, since $K^2 = -1$,

$$\begin{aligned} &g(Y, -\tfrac{1}{2}K(\hat{\nabla}_Z K)KZ + \tfrac{1}{2}K(\hat{\nabla}_{KX}K)Z - \tfrac{1}{2}(\hat{\nabla}_{KZ}K)KX \\ &+ \tfrac{1}{2}(\hat{\nabla}_{KX}K)KZ + \tfrac{1}{2}(\hat{\nabla}_Z K)X - \tfrac{1}{2}(\hat{\nabla}_X K)Z - \tfrac{1}{2}K(\hat{\nabla}_{KZ}K)X + \tfrac{1}{2}K(\hat{\nabla}_X K)KZ) = 0. \end{aligned}$$

On the other hand, the sum of the 6th, 8th, 14th and 16th terms is also zero, since it is analogous to the earlier, changing X and Y. Thus, it rests only the sum of the 1st, 5th, 9th and 13th terms, which is precisely the right-hand side member of (5.4).

REFERENCES

[1] R. Bott, *Lectures on characteristic classes and foliations*, Lecture Notes in Math. 279, pp. 1–94, Springer-Verlag, 1972.

[2] J. M. Hernando, P. M. Gadea and A. Montesinos, *G-structures defined by tensor fields of electromagnetic type* (to appear).

[3] V. Hlavatý, *Geometry of Einstein's Unified Field Theory*, P. Noordhoff, 1958.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*. II, Intersc. Publ., 1969.

[5] R. S. Mishra, *Structures in electromagnetic tensor fields*, Tensor N. S. 30 (1976), pp. 145–156.

[6] M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Japan J. Math. 26 (1976), pp. 43–77.

[7] E. Reyes, A. Montesinos and P. M. Gadea, *Connections making parallel a metric ($J^4 = 1$)-structure*, An. Şti. Univ. "Al. I. Cuza" 28 (1982), pp. 49–54.

[8] I. Vaisman, *Variétés riemanniennes feuilletées*, Czechoslovak Math. J. 21 (96) (1971), pp. 46–75.

[9] – *From the geometry of Hermitian foliate manifolds*, Bull. Math. Soc. Sci. Math. R. S. Roumanie 17 (1965), pp. 71–100.
[10] K. Yano, *On a structure f satisfying* $f^3+f=0$, Tech. Rep. No. 12 (1961), Univ. of Washington.

DEPARTAMENTO DE GEOMETRÍA
FACULTAD DE CIENCIAS
VALLADOLID, SPAIN
INSTITUTO JORGE JUAN
C.S.I.C., MADRID, SPAIN

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE MATEMÁTICAS
BURJASOT (VALENCIA), SPAIN

Reçu par la Rédaction le 30. 5. 1985