COLLOQUIUM MATHEMATICUM

XXVI DÉDIÉ À M. WŁADYSŁAW ŚLEBODZIŃSKI 1972

A CERTAIN MODIFICATION OF HARDY'S INEQUALITY

BY

B. FLORKIEWICZ AND A. RYBARSKI (WROCŁAW)

Let $\{\lambda_n\}$ be an arbitrary fixed sequence of real positive numbers. The aim of the present paper is to estimate the optimal constant μ in the inequality

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_1 a_1+\ldots+\lambda_n a_n}{\lambda_1+\ldots+\lambda_n}\right)^2 \leqslant \mu \sum_{n=1}^{\infty} a_n^2 \tag{1}$$

which should hold for every sequence $\{a_n\}$ of real numbers. Namely, we shall prove the following theorems:

THEOREM 1. *If a sequence $\{\lambda_n\}$ is non-decreasing, then $\mu \leqslant 4$.*

THEOREM 2. *Let $\sigma_n = \lambda_1+\ldots+\lambda_n$. If $\sigma_n \to +\infty$ for $n \to \infty$ and $\sum_{n=1}^{\infty} \sigma_n^{-1} = +\infty$, then $\mu \geqslant 4$.*

Assumptions of both theorems are satisfied by the sequence $\lambda_n = 1$, $n = 1, 2, \ldots$, hence we have $\mu = 4$. In this case (1) turns out into Hardy's inequality (cf. [3]). One also has the value $\mu = 4$ in the case of the sequences $\lambda_n = 1-q^n$, where $0 < q < 1$, $\lambda_n = 1+\ln n$, and some others (cf. [2]).

Let us introduce the notation

$$\varrho_n = \sum_{k=n}^{\infty} \sigma_k^{-2} \quad \text{and} \quad \omega_n = \lambda_n^2 \varrho_n + 2\lambda_n^2 \varrho_n^2 \sigma_{n-1}^2$$

for $n = 1, 2, \ldots$, where $\sigma_0 = 0$.

The proof of theorem 1 will be based on the inequality

$$\mu \leqslant 2 \sup_n \omega_n. \tag{2}$$

In order to prove this inequality, put

$$A_n = \lambda_1 a_1+\ldots+\lambda_n a_n$$

and

$$l_n = \sum_{k=1}^{n}\left(\frac{A_k}{\sigma_k}\right)^2 \quad \text{and} \quad \varrho_k^n = \sum_{i=k}^{n} \frac{1}{\sigma_i^2} \quad \text{for } 1 \leqslant k \leqslant n.$$

Now the identity

$$l_n = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 l_n}{\partial a_i \partial a_j} a_i a_j$$

can be easily transformed into the form

$$l_n = \sum_{k=1}^{n} \lambda_k^2 \varrho_k^n a_k^2 + 2\sum_{k=2}^{n} \lambda_k \varrho_k^n a_k A_{k-1},$$

whence, applying Schwarz's inequality, we get the estimation

$$l_n \leqslant \sum_{k=1}^{n} \lambda_k^2 \varrho_k^n a_k^2 + 2 l_n^{1/2} \Big\{\sum_{k=1}^{n} \lambda_k^2 (\varrho_k^n)^2 \sigma_{k-1}^2 a_k^2\Big\}^{1/2}$$

from which it follows that

$$l_n \leqslant 2\sum_{k=1}^{n} [\lambda_k^2 \varrho_k^n + 2\lambda_k^2 (\varrho_k^n)^2 \sigma_{k-1}^2] a_k^2.$$

Now, if $n \to +\infty$, then

$$\sum_{k=1}^{\infty} \left(\frac{A_k}{\sigma_k}\right)^2 \leqslant 2\sum_{k=1}^{\infty} \omega_k a_k^2$$

which already demonstrates the validity of (2). It seems worth-while to notice that in this proof the only property we have used is that of $\sigma_n \neq 0$.

Proof of theorem 1. If a sequence $\{\lambda_n\}$ is non-decreasing, then $\sigma_{n+k} \geqslant \sigma_{n-1} + (k+1)\lambda_n$, whence

$$\varrho_n = \sum_{k=0}^{\infty} \frac{1}{\sigma_{n+k}^2} \leqslant \sum_{k=1}^{\infty} (\sigma_{n-1} + k\lambda_n)^{-2}$$

and, therefore, $\lambda_n^2 \varrho_n \leqslant h(s_n)$, where $s_n = \sigma_{n-1}/\lambda_n$ and $h(s) = \sum_{k=1}^{\infty} (s+k)^{-2}$. This implies the estimation $\omega_n \leqslant \varphi(s_n)$, where $\varphi(s) = h(s) + 2s^2 h^2(s)$ and $\omega_n = \lambda_n^2 \varrho_n [1 + 2(\lambda_n^2 \varrho_n) s_n^2]$. If we shall show that for $s \geqslant 0$ there is $\varphi(s) \leqslant 2$, then $\omega_n \leqslant 2$, which will prove theorem 1 in virtue of (2).

Proof of the estimation $\varphi(s) \leqslant 2$. Using the identity

$$\frac{1}{s+1} = \sum_{k=1}^{\infty} \frac{1}{(s+k)(s+k+1)}$$

we find

$$\frac{1}{s+1}-h(s)+\frac{1}{(s+1)^2}$$
$$=\sum_{k=1}^{\infty}\frac{1}{(s+k)(s+k+1)^2}>\int_1^{\infty}\frac{dx}{(s+x)(s+x+1)^2}>\frac{1}{2(s+2)^2},$$

whence

$$h(s)\leqslant\frac{1}{s+1}+\frac{1}{(s+1)^2}-\frac{1}{2(s+2)^2}\qquad\text{for } s\geqslant 0.$$

Performing elementary calculations, we get the estimation

$$\varphi(s)\leqslant 2-2(s+1)(s+2)^{-4}-(s+1)^{-4}(s+2)^{-1}(s^4+2s^3+5s^2+6s)\leqslant 2.$$

Proof of theorem 2. For a given $N\geqslant 2$ take the sequence $\{a_n\}$ defined by the formulae $a_1=0$ and $a_n=1/\sqrt{\sigma_{n-1}}$ for $2\leqslant n\leqslant N$, and $a_n=0$ for $n>N$. Consider the numbers

$$\eta_N=\sum_{n=1}^{N}\left(\frac{A_n}{\sigma_n}\right)^2\left(\sum_{n=1}^{N}a_n^2\right)^{-1}.$$

By the definition of μ, we have $\mu\geqslant\eta_N$. We shall show that $\liminf_{N\to\infty}\eta_N\geqslant 4$, and so that $\mu\geqslant 4$ which will prove the theorem.

The formulae defining A_n's give here $A_1=0$ and

$$A_n=\frac{\lambda_2}{\sqrt{\sigma_1}}+\ldots+\frac{\lambda_n}{\sqrt{\sigma_{n-1}}}\geqslant\int_{\sigma_1}^{\sigma_n}\frac{dx}{\sqrt{x}}=2(\sqrt{\sigma_n}-\sqrt{\sigma_1})\qquad\text{for } 2\leqslant n\leqslant N.$$

Hence, it follows that

$$\eta_N\geqslant 4\sum_{n=1}^{N-1}\frac{1}{\sigma_n}\left(1-\sqrt{\frac{\sigma_1}{\sigma_n}}\right)^2\left(\sum_{n=1}^{N-1}\frac{1}{\sigma_n}\right)^{-1}.\tag{3}$$

We have assumed that the series $\sum_{n=1}^{\infty}\sigma_n^{-1}$ is divergent. It follows then from Stolz's lemma (cf. [1]) that the right-hand side of inequality (3) tends to

$$4\lim_{n\to\infty}\left(1-\sqrt{\frac{\sigma_1}{\sigma_n}}\right)^2=4\qquad\text{if } N\to\infty.$$

This completes the proof.

Final remarks. Let us consider the operation

$$S\{a_n\}=\left\{\frac{a_1+\ldots+a_n}{n}\right\}$$

defined in the space l_2. Hardy's inequality states that the linear operation S is bounded and $\|S\| = 2$. One can regard our results as an estimation of the norm of l_2 for somewhat more general operation

$$S\{a_n\} = \left\{\frac{\lambda_1 a_1 + \ldots + \lambda_n a_n}{\lambda_1 + \ldots + \lambda_n}\right\}.$$

Under appropriate assumptions we get the same result $\|S\| = 2$.

Although there are nowadays several proofs of Hardy's inequality (see [3]), one can treat our considerations as one another proof. However, the case of Hardy's inequality turning to the equality must be dealt with separately. It is known that the case takes place only if $a_n = 0$, $n = 1, 2, \ldots$ Is the same thesis valid for inequality (1)? (**P 817**)

REFERENCES

[1] Т. М. Фихтенгольц, *Курс дифференциального и интегрального исчисления*, том I, Москва-Ленинград 1948.

[2] B. Florkiewicz and A. Rybarski, *On an integral inequality connected with Hardy's inequality (III)*, Colloquium Mathematicum (in print).

[3] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, London 1951.

NSTITUTE OF ORGANIZATION AND ECONOMICS, TECHNICAL UNIVERSITY OF WROCŁAW
IINSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 7. 10. 1971