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A GENERALIZED ŁOŚ ULTRAPRODUCT THEOREM

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1. Introduction. This paper is a sequel of [1]. We shall prove here a purely semantical version of the Compactness Theorem: every set of sentences has a model iff every finite subset has a model. To prove it we generalize the notion of an ultraproduct and the well-known Łoś Theorem to weak models for L_{Q_1}. From this the Compactness Theorem easily follows. It must be said that this theorem also follows from the results of Keisler ([2], Theorem 3.15.1). Concerning the Löwenheim-Skolem Theorem, it follows from a counterexample that the Downward Löwenheim-Skolem Theorem holds only in the case corresponding to the definition of $\subseteq_1$ and that the Upward Löwenheim-Skolem Theorem holds in the cases corresponding to the definitions of $\subseteq_1$, $\subseteq_3$ and $\subseteq_4$. Finally, we make an attempt to describe the sets of sentences preserved under the operation of forming extensions. Unfortunately, we did not succeed in giving a full description of these sets.

2. Compactness Theorem and Löwenheim-Skolem Theorems. It follows from Keisler [2] that there exists an axiomatization for L_Q such that every sentence which is true in all structures for L_Q is deducible in L_Q and conversely. Namely, let Δ be a set of axioms for L including the axioms for equality. Let Δ_Q be

$$\Delta \cup \{\forall x(\varphi \leftrightarrow \psi) \rightarrow (Q\, x\, \varphi \leftrightarrow Q\, x\, \psi),\ (Q\, x)\, \varphi(x) \leftrightarrow (Q\, y)\, \varphi(y) \quad \text{if } y \text{ is a variable which is not free in } \varphi(x)\}.$$

As deduction rules we take the modus ponens and the generalization rule.

THEOREM 2.1 (Keisler [2], Theorem 3.15.1). *Let L_Q be countable and let Σ be a set of sentences in L_Q. The following are equivalent:*

(i) *Σ is consistent relative to Δ_Q.*

(ii) *Σ has a model.*

COROLLARY (Compactness Theorem). *Let L_Q be countable and let Σ be a set of sentences in L_Q. Σ has a model iff every finite subset of Σ has a model.*

Remark. Both these theorems still hold if we omit the assumption that L_Q is countable.

We are going to prove the Compactness Theorem without using an axiomatization. The method is to make an ultraproduct construction and to prove the Łoś Theorem for it.

Let $(\mathfrak{A}_i, p_i)_{i\in I}$ be a collection of structures for L_Q and let F be an ultrafilter over I. The *ultraproduct*, written as $[\prod_{i\in I}(\mathfrak{A}_i, p_i)]|F$, is defined as follows:

$$\Big[\prod_{i\in I}(\mathfrak{A}_i, p_i)\Big]\Big|F = (\mathfrak{A}, p), \quad \text{where } \mathfrak{A} = \Big[\prod_{i\in I}\mathfrak{A}_i\Big]\Big|F,$$

and $p \subseteq S(\prod A_i|F)$ is such that $x \in p$ if there are $x_i \subseteq A_i$ with

$$\{i : x_i \in p_i\} \in F \quad \text{and} \quad x = \Big\{\bar{a} \in \prod A_i|F\colon \{i \in I\colon a(i) \in x_i\} \in F\Big\}.$$

This definition of p seems to be somewhat involved. A nicer one would be the following: $x \in p$ if there are $x_i \in p_i$ such that $x = \prod_{i\in I} x_i|F$, but we have to take into account the possibility $p_i = \emptyset$ or $\emptyset \in p_i$.

We state the following easy proposition without proof:

PROPOSITION 1. (a) $\{i \in I\colon \emptyset \in p_i\} \in F$ *iff* $\emptyset \in p$.

(b) $\{i \in I\colon p_i = \emptyset\} \in F$ *iff* $p = \emptyset$.

This proposition we need in the following theorem, which is a generalization of the Łoś Theorem.

THEOREM 2.2. *For all formulas φ in L_Q and for all $\bar{a}_1, \ldots, \bar{a}_n \in \prod_{i\in I} A_i|F$,*

$$(\mathfrak{A}, p) \models \varphi[\bar{a}_1, \ldots, \bar{a}_n] \quad \textit{iff} \quad \{i \in I\colon (\mathfrak{A}_i, p_i) \models \varphi[a_1(i), \ldots, a_n(i)]\} \in F.$$

Proof. We prove this by induction on the construction of φ. Only the case $\varphi = (Q\, v_m)\psi$ is not trivial. Suppose

$$\{i \in I\colon (\mathfrak{A}_i, p_i) \models (Q\, v_m)\,\psi[a_1(i), \ldots, a_n(i)]\} \in F,$$

and let

$$B_i = \{a \in A_i\colon (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a, \ldots, a_n(i)]\}.$$

Then we have $\{i \in I\colon B_i \in p_i\} \in F$. Let $X = \{i \in I\colon B_i = \emptyset\}$. We consider two cases (for $X \in F$ and $X \notin F$).

(a) $X \in F$. Then we have

$$\big\{i \in I\colon (\mathfrak{A}_i, p_i) \models \neg(\exists\, v_m)\psi[a_1(i), \ldots, a_n(i)]\big\} \in F \quad \text{and} \quad \{i \in I\colon \emptyset \in p_i\} \in F,$$

and so, by Proposition 1, $\emptyset \in p$. We have, by the induction hypothesis,

$$(\mathfrak{A}, p) \models \neg\exists\, v_m\psi[\bar{a}_1, \ldots, \bar{a}_n].$$

Since $\emptyset \in p$ and $(\mathfrak{A}, p) \models \neg \exists v_m \psi[\bar{a}_1, \ldots, \bar{a}_n]$, we have

$$(\mathfrak{A}, p) \models (Q\, v_m)\, \psi[\bar{a}_1, \ldots, \bar{a}_n],$$

which we had to prove.

(b) $X \notin F$. Then $I - X \in F$, i.e. $\{i \in I\colon B_i \neq \emptyset\} \in F$. For $B_i \neq \emptyset$, let $x_i = B_i$; otherwise let $x_i = A_i$. Then, if we put

$$Y = \{i \in I\colon x_i = B_i \text{ and } x_i \in p_i\},$$

we have $Y \in F$.

CLAIM. *If* $x = \{\bar{a}\colon \{i\colon a(i) \in x_i\} \in F\}$, *then* $x \in p$ *and*

$$x = \Big\{\bar{a} \in \prod_{i \in I} A_i \,|\, F\colon (\mathfrak{A}, p) \models \psi[\bar{a}_1, \ldots, \bar{a}, \ldots, \bar{a}_n]\Big\}.$$

Proof. It follows immediately from the definition of p that $x \in p$. So we have to prove that

$$(\mathfrak{A}, p) \models \psi[\bar{a}_1, \ldots, \bar{a}, \ldots, \bar{a}_n] \quad \text{iff} \quad \{i\colon a(i) \in x_i\} \in F \quad \text{for all } \bar{a} \in \prod_{i \in I} A_i | F.$$

Let $\bar{a}$ be such that

$$(\mathfrak{A}, p) \models \psi[\bar{a}_1, \ldots, \bar{a}, \ldots, \bar{a}_n],$$

and let

$$Z = \{i \in I\colon (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)]\}.$$

Then our hypothesis on ψ gives $Z \in F$. So, if $i \in Z$, then

$$(\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)],$$

whence $a(i) \in B_i$. Consequently, we have $a(i) \in x_i$. So

$$Z \subseteq \{i \in I\colon a(i) \in x_i\} \quad \text{and} \quad Z \in F.$$

This implies that $\{i \in I\colon a(i) \in x_i\} \in F$.

Let $\bar{a} \in \prod_{i \in I} A_i | F$ be given such that $\{i \in I\colon a(i) \in x_i\} \in F$, and let $S = \{i \in I\colon a(i) \in x_i\}$. If $i \in Y$, then $x_i = B_i$ and $x_i \in p_i$. If $i \in S$, then $a(i) \in x_i$. So, if $i \in Y \cap S$, then

$$a(i) \in x_i \in p_i \quad \text{and} \quad (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)].$$

From this it follows that

$$S \cap Y \subseteq \{i\colon (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)]\}.$$

Since $S \cap Y \in F$, we have

$$\{i\colon (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)]\} \in F.$$

The induction hypothesis on ψ gives $(\mathfrak{A}, p) \models \psi[\bar{a}_1, \ldots, \bar{a}, \ldots, \bar{a}_n]$. So our claim is proved, and now we may conclude with

$$(\mathfrak{A}, p) \models (Q\, v_m)\, \psi[\bar{a}_1, \ldots, \bar{a}_n],$$

the proof of which was required.

Now suppose $(\mathfrak{A}, p) \models (Q\, v_m)\, \psi[\bar{a}_1, \ldots, \bar{a}_n]$. This means that there are $x_i \subseteq A_i$ such that if $Y = \{i \in I\colon x_i \in p_i\}$, then $Y \in F$ and

$$(\mathfrak{A}, p) \models \psi[\bar{a}_1, \ldots, \bar{a}, \ldots, \bar{a}_n] \quad \text{iff} \quad \{i \in I\colon a(i) \in x_i\} \in F.$$

Applying the induction hypothesis on ψ, we have

$$(*) \quad \{i \in I\colon (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)]\} \in F \quad \text{iff } \{i \in I\colon a(i) \in x_i\} \in F.$$

We have to prove that

$$\{i \in I\colon (\mathfrak{A}_i, p_i) \models (Q\, v_m)\, \psi[a_1(i), \ldots, a_n(i)]\} \in F.$$

Let $B_i = \{a \in A_i\colon (\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a, \ldots, a_n(i)]\}$ and $X = \{i \in I\colon B_i = \emptyset\}$. We consider two cases (for $X \in F$ and $X \notin F$).

(a) $X \in F$. In this case we have

$$\{i \in I\colon (\mathfrak{A}_i, p_i) \models \neg \exists\, v_m\, \psi[a_1(i), \ldots, a_n(i)]\} \in F.$$

Then it follows from $(*)$ that, for no $a \in \prod_{i \in I} A_i$, $\{i \in I\colon a(i) \in x_i\} \in F$. This implies that $\{i \in I\colon x_i = \emptyset\} \in F$, and so, in view of $Y \in F$, we have $\{i \in I\colon \emptyset \in p_i\} \in F$. If

$$\{i \in I\colon (\mathfrak{A}_i, p_i) \models \neg \exists\, v_m\, \psi[a_1(i), \ldots, a_n(i)]\} \in F \quad \text{and} \quad \{i \in I\colon \emptyset \in p_i\} \in F,$$

then

$$\{i \in I\colon \emptyset \in p_i \text{ and } (\mathfrak{A}_i, p_i) \models \neg \exists\, v_m \psi[a_1(i), \ldots, a_n(i)]\} \in F.$$

Hence we may conclude the formula

$$\{i \in I\colon (\mathfrak{A}_i, p_i) \models (Q\, v_m)\, \psi[a_1(i), \ldots, a_n(i)]\} \in F,$$

the proof of which was required.

(b) $X \notin F$. We have to prove that $\{i \in I\colon B_i \in p_i\} \in F$. We shall prove

$$(**) \qquad \{i \in I\colon B_i = x_i\} \in F.$$

If $(**)$ is proved, then we have $Y \cap \{i \in I\colon B_i = x_i\} \in F$, which will complete the proof.

Proof of $(**)$. Let

$$X_1 = \{i \in I - X\colon x_i - B_i \neq \emptyset\},$$

$$X_2 = \{i \in I - X\colon x_i \subsetneq B_i\} \quad \text{and} \quad X_3 = \{i \in I - X\colon x_i = B_i\}.$$

We shall prove that $X_3 \in F$. This is enough, because then we have $X_3 \subseteq \{i \in I: B_1 = x_i\}$, and so $\{i \in I: B_i = x_i\} \in F$. Take an $a \in \prod_{i \in I} A_i$ such that

(i) if $i \in X$, then $a(i)$ is arbitrary;

(ii) $a(i) \in x_i - B_i$ for all $i \in X_1$;

(iii) $a(i) \in B_i - x_i$ for all $i \in X_2$;

(iv) $a(i) \in x_i = B_i$ for all $i \in X_3$.

We have

$$(\mathfrak{A}_i, p_i) \models \psi[a_1(i), \ldots, a(i), \ldots, a_n(i)] \quad \text{iff} \quad a(i) \in B_i \quad \text{iff} \quad i \in X_2 \cup X_3$$

and

$$a(i) \in x_i \quad \text{iff} \quad i \in Z \cup X_1 \cup X_3, \quad \text{where } Z = \{i \in X: a(i) \in x_i\}.$$

From $(*)$ it follows that $X_2 \cup X_3 \in F$ iff $Z \cup X_1 \cup X_3 \in F$. If $X_2 \cup X_3 \notin F$, then also $Z \cup X_1 \cup X_3 \notin F$. Then, since F is an ultrafilter, we have

$$(I - X) \cup Z = X_2 \cup X_3 \cup Z \cup X_1 \cup X_3 \notin F.$$

But $X \notin F$, and so $I - X \in F$, whence $(I - X) \cup Z \in F$.

Thus the only possibility is $X_2 \cup X_3 \in F$. Then also $Z \cup X_1 \cup X_3 \in F$, and from this it follows that

$$X_3 = (X_2 \cup X_3) \cap (Z \cup X_1 \cup X_3) \in F.$$

This proves $(**)$, and so the proof of the theorem is complete.

COMPACTNESS THEOREM. *Let Σ be a set of sentences in L_Q such that every finite subset of Σ has a model. Then Σ has a model.*

Proof goes in a standard way with the ultraproduct construction.

Let $(\mathfrak{A}, p)$ be a structure for L_Q, and d_F the diagonal embedding of A into A^I_F, where I is a set and F an ultrafilter over I. From the Łoś Theorem it follows that $d_F(\mathfrak{A}, p) \prec (\mathfrak{A}, p)^I_F$. In our case we have

$$d_F(\mathfrak{A}, p) \prec_1 (\mathfrak{A}, p)^I_F.$$

To see it, let $x \in p$ and $y = \{\bar{a}: \{i: a(i) \in x\} \in F\}$. Then $y \cap d_F{*}A = x$ and $y \in q$, where $(\mathfrak{A}, p)^I_F = (\mathfrak{A}^I_F, q)$.

If $i \in \{2, 3, 4\}$, then, in general, we do not have $d_F(\mathfrak{A}, p) \prec_i (\mathfrak{A}, p)^I_F$.

Examples. (a) $i = 2$. Let $\mathfrak{A} = \langle \mathbf{Z}, < \rangle$ and $p = \{\{n \in \mathbf{Z}: n < m\}: m \in \mathbf{Z}\}$. Let $I = \omega$ and let F be an ultrafilter over I that contains the sets $\{n \in \omega: n \geqslant k\}$ for all $k \in \omega$. If $f \in \mathbf{Z}^I$ is such that $f(i) = i$ for all $i \in I$, then

$$(\mathfrak{A}, p)^I_F \models Q\, v_0\, (v_0 < v_1)[\bar{f}],$$

which is easy to see by Theorem 2.2. If

$$x = \{\bar{g} \in \mathbf{Z}^I_F: (\mathfrak{A}, p)^I_F \models (v_0 < v_1)[\bar{g}, \bar{f}]\},$$

then $x \in q$, where $(\mathfrak{A}, p)^I_F = (\mathfrak{A}^I_F, q)$.

Now consider the set $\{\bar{g} \in x\colon g$ is a constant function$\}$. This set is equal to $\{\bar{g} \in Z_F^I\colon g$ is a constant function$\}$, and so it is not equal to $d_F * y$ for any $y \in p$. This shows that $d_F(\mathfrak{A}, p) \prec_2 (\mathfrak{A}, p)_F^I$ is not the case.

(b) $i = 3$ or $i = 4$. Let $\mathfrak{A} = \langle\omega\rangle$ and $p = \{\omega\}$. Let $I = \omega$ and let F be a non-principal ultrafilter over I. If we write $(\mathfrak{A}, p)_F^I = (\mathfrak{A}_F^I, q)$, then q has obviously only one element. This element has cardinality $|\omega_F^\omega| = 2^\omega$. This shows that $d_F * \omega \notin q$, and so $d_F(\mathfrak{A}, p) \prec_i (\mathfrak{A}, p)_F^I$ is not the case for $i \in \{3, 4\}$.

Remark. In the case of non-measurable ultrapowers it is possible to extend our definition of ultrapower to get the following result: the diagonal structure is an elementary 4-submodel of the ultrapower. To show it we proceed as follows. Let $(\mathfrak{B}, q)$ be a non-measurable ultrapower of $(\mathfrak{A}, p)$ defined as before. Let $q^* = q_0 \cup p$, where q_0 is minimal with respect to the q-interpretation of Q in $\mathfrak{B}$. Now we have $(\mathfrak{A}, p) \prec_4 (\mathfrak{B}, q^*)$. This follows from the fact that in the case of a non-measurable ultrapower, a subset of the diagonals is definable in the ultrapower iff it is finite. Thus $(\mathfrak{B}, q)$ is good with respect to $(\mathfrak{A}, p)$ (for the definition of goodness see p. 167).

Now we shall prove the Downward Löwenheim-Skolem Theorem for $\prec_1$ and the Upward Löwenheim-Skolem Theorem for $\prec_1$, $\prec_3$ and $\prec_4$. A counterexample will show that in the other cases the Löwenheim--Skolem Theorem does not hold.

THEOREM 2.3. *Let $(\mathfrak{A}, p)$ be a structure for L_Q such that $|L_Q| \leqslant |A|$. Suppose C is a subset of A and $\varkappa$ is a cardinal such that $|L_Q| + |C| \leqslant \varkappa \leqslant |A|$. Then there is a structure $(\mathfrak{B}, q)$ such that $C \subseteq B$, $|B| = \varkappa$ and $(\mathfrak{B}, q) \prec_1 (\mathfrak{A}, p)$.*

Proof. The structure $(\mathfrak{B}, q)$ in Lemma 5.1 in Keisler [2] works, since q is minimal. (For the definition of minimality see [1].)

THEOREM 2.4 (Upward Löwenheim-Skolem Theorem for $\prec$). *Let $(\mathfrak{A}, p)$ be given such that A is infinite. Let $\varkappa$ be a cardinal such that $\varkappa \geqslant |L_Q| + |A|$. Then there is a structure $(\mathfrak{B}, q)$ with $(\mathfrak{A}, p) \prec (\mathfrak{B}, q)$ and $|B| = \varkappa$.*

Proof runs in a standard way using the Compactness Theorem and Theorem 2.3.

COROLLARY 1 (Upward Löwenheim-Skolem Theorem for $\prec_1$). *Let $(\mathfrak{A}, p)$ be given such that p is minimal and $|A| \geqslant \omega$, and let $\varkappa$ be a cardinal such that $|L_Q| + |A| \leqslant \varkappa$. Then there is a structure $(\mathfrak{B}, q)$ such that $|B| = \varkappa$ and $(\mathfrak{A}, p) \prec_1 (\mathfrak{B}, q)$.*

Proof follows immediately from Theorem 2.4 and the remark about $\prec_1$ and $\prec$ in the beginning of Section 6 in [1].

COROLLARY 2. *If Σ is a theory in L_Q with a model of cardinality not less than $|L_Q|$, then Σ has models of each cardinality not less than $|L_Q|$.*

Definition. Let $(\mathfrak{A}, p) \prec (\mathfrak{B}, q)$. Then $(\mathfrak{B}, q)$ is said to be *good with respect to* $(\mathfrak{A}, p)$ if the following condition is satisfied:

For all $\varphi \in L_Q$ and $b_1, \ldots, b_n \in B$, if $\{c \in A: (\mathfrak{B}, q) \models \varphi[b_1, \ldots, c, \ldots, b_n]\}$ is infinite, then

$$\{c \in B: (\mathfrak{B}, q) \models \varphi[b_1, \ldots, c, \ldots, b_n]\} \nsubseteq A.$$

LEMMA 1. *Let $(\mathfrak{A}, p)$ be given such that $|A| \geqslant \omega$, and let $\varkappa$ be a cardinal such that $\varkappa \geqslant |L_Q| + |A|$. Then there is a structure $(\mathfrak{B}, q)$ such that $|B| = \varkappa$, $(\mathfrak{A}, p) \prec (\mathfrak{B}, q)$ and $(\mathfrak{B}, q)$ is good with respect to $(\mathfrak{A}, p)$.*

Proof. We construct a sequence $(\mathfrak{B}_n, q_n)_{n \in \omega}$ such that $(\mathfrak{B}_n, q_n) \prec (\mathfrak{B}_{n+1}, q_{n+1})$, and $(\mathfrak{A}, p) = (\mathfrak{B}_0, q_0)$ is the structure defined in Theorem 6.4 of [1]. Let $L_{Q,0} = L_Q \cup \{c_a: a \in A\}$. Then $|L_{Q,0}| \leqslant \varkappa$. Put $(\mathfrak{B}_0, q_0) = (\mathfrak{A}, p)$ and let $(\mathfrak{B}_n, q_n)$ be constructed. Let

$$Z_n = \big\{\langle \varphi, a_1, \ldots, a_n\rangle: \varphi \in L_Q, a_1, \ldots, a_n \in B_n \text{ and } \{b \in A: (\mathfrak{B}_n, q_n) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \text{ is infinite}\big\}.$$

Let $L_{Q,n+1}$ be the language which we obtain from $L_{Q,n}$ by adding a new individual constant $d_{n,\bar{x}}$ for $\bar{x} \in Z_n$. Assuming $|L_{Q,n}| \leqslant \varkappa$, we have $|L_{Q,n+1}| \leqslant \varkappa$. Let

$$\Sigma_n = \{\varphi(c_{a_1}, \ldots, d_{n,\bar{x}}, \ldots, c_{a_m}): \bar{x} \in Z_n, \bar{x} = \langle \varphi, a_1, \ldots, a_n\rangle\} \cup \cup \{d_{n,\bar{x}} \neq c_a: a \in A, \bar{x} \in Z_n\} \cup \mathrm{Th}(\mathfrak{B}_n, q_n, b)_{b \in B_n}.$$

Since every finite subset of Σ_n has a model, so Σ_n has a model $(\mathfrak{B}_{n+1}, q_{n+1})$ of cardinality $\varkappa$.

If we identify $c_a^{\mathfrak{B}_{n+1}}$ with $c_a^{\mathfrak{B}_n}$ for each $a \in A$ and $d_{i,\bar{x}}^{\mathfrak{B}_{n+1}}$ with $d_{i,\bar{x}}^{\mathfrak{B}_n}$ for each $i \leqslant n-1$ and $\bar{x} \in Z_i$, then we have

$$(\mathfrak{B}_n, q_n) \prec (\mathfrak{B}_{n+1}, q_{n+1}).$$

$(\mathfrak{B}_n, q_n)_{n \in \omega}$ is a simple elementary chain. Let $(\mathfrak{B}, q)$ be the structure as defined in Theorem 6.4 of [1]. Then $|B| = \varkappa$ and $(\mathfrak{A}, p) \prec (\mathfrak{B}, q)$.

CLAIM. *$(\mathfrak{B}, q)$ is good with respect to $(\mathfrak{A}, p)$.*

Proof. Let $\{c \in A: (\mathfrak{B}, q) \models \varphi[a_1, \ldots, c, \ldots, a_m]\}$ be infinite. There is an $n \in \omega$ such that $\langle \varphi, a_1, \ldots, a_m\rangle = \bar{x} \in Z_n$. Then we have

$$(\mathfrak{B}_{n+1}, q_{n+1}) \models \varphi[a_1, \ldots, d_{n,\bar{x}}^{\mathfrak{B}_{n+1}}, \ldots, a_m],$$

and so

$$(\mathfrak{B}, q) \models \varphi[a_1, \ldots, d_{n,\bar{x}}^{\mathfrak{B}_{n+1}}, \ldots, a_m].$$

Since $d_{n,\bar{x}}^{\mathfrak{B}_{n+1}} \notin A$, we have

$$\{c \in B: (\mathfrak{B}, q) \models \varphi[a_1, \ldots, c, \ldots, a_n]\} \nsubseteq A.$$

This proves the claim, and so the lemma is proved.

PROPOSITION 2. *Let* $(\mathfrak{A}, p) \prec (\mathfrak{B}, q)$ *and let* $x \subseteq A$ *be finite. Then* $x \in p$ *iff* $x \in q$.

Proof. Let $x = \{a_1, \ldots, a_n\} \subseteq A$. Then

$$(\mathfrak{A}, p) \models (Qx)(x = v_1 \vee \ldots \vee x = v_n)\,[a_1, \ldots, a_n]$$
$$\text{iff } (\mathfrak{B}, q) \models (Qx)(x = v_1 \vee \ldots \vee x = v_n)\,[a_1, \ldots, a_n],$$

which completes the proof.

THEOREM 2.5 (Upward Löwenheim-Skolem Theorem for $\prec_3$ and $\prec_4$). *Let* $(\mathfrak{A}, p)$ *be given such that* $|A| \geqslant \omega$. *Let* $\varkappa$ *be a cardinal such that* $\varkappa \geqslant |L_Q| + |A|$. *Then there is a structure* $(\mathfrak{B}, q)$ *with* $|B| = \varkappa$ *and* $(\mathfrak{A}, p) \prec_4 (\mathfrak{B}, q)$, *and so* $(\mathfrak{A}, p) \prec_1 (\mathfrak{B}, q)$ *and* $(\mathfrak{A}, p) \prec_3 (\mathfrak{B}, q)$.

Proof. There is a $(\mathfrak{B}, q')$ such that $|B| = \varkappa$, $(\mathfrak{A}, p) \prec (\mathfrak{B}, q')$ and $(\mathfrak{B}, q')$ is good with respect to $(\mathfrak{A}, p)$. Let q'' be minimal with respect to the q'-interpretation of Q in $\mathfrak{B}$. Then (B, q'') is good with respect to $(\mathfrak{A}, p)$. Let $q = q'' \cup p$.

CLAIM 1. *For all* $\varphi \in L_Q$ *and* $b_1, \ldots, b_n \in B$,

$$(\mathfrak{B}, q'') \models \varphi[b_1, \ldots, b_n] \quad \textit{iff} \quad (\mathfrak{B}, q) \models \varphi[b_1, \ldots, b_n].$$

Proof runs by induction on the construction of φ. Only the case $\varphi = (Qv_m)\psi$ is not trivial.

Suppose $(\mathfrak{B}, q'') \models (Q\, v_m)\, \psi[b_1, \ldots, b_n]$. Then

$$\{c \in B\colon (\mathfrak{B}, q'') \models \psi[b_1, \ldots, c, \ldots, b_n]\} \in q''.$$

The induction hypothesis on ψ gives

$$\{c \in B\colon (\mathfrak{B}, q) \models \psi[b_1, \ldots, c, \ldots, b_n]\} \in q'',$$

and so

$$\{c \in B\colon (\mathfrak{B}, q) \models \psi[b_1, \ldots, c, \ldots, b_n]\} \in q,$$

whence

$$(\mathfrak{B}, q) \models (Q\, v_m)\, \psi[b_1, \ldots, b_n].$$

Suppose $(\mathfrak{B}, q) \models (Q\, v_m)\, \psi[b_1, \ldots, b_n]$. Then

$$\{c \in B\colon (\mathfrak{B}, q) \models \psi[b_1, \ldots, c, \ldots, b_n]\} \in q'' \cup p.$$

Let $\{c \in A\colon (\mathfrak{B}, q'') \models \psi[b_1, \ldots, c, \ldots, b_n]\}$ be finite. Then, by Proposition 2,

$$(\mathfrak{B}, q'') \models Q\, v_m\, \psi[b_1, \ldots, b_n].$$

If $\{c \in A\colon (\mathfrak{B}, q'') \models \psi[b_1, \ldots, c, \ldots, b_n]\}$ is infinite, then

$$\{c \in B\colon (\mathfrak{B}, q'') \models \psi[b_1, \ldots, c, \ldots, b_n]\} \nsubseteq A,$$

and so

$$\{c \in B\colon (\mathfrak{B}, q'') \models \psi[b_1, \ldots, c, \ldots, b_n]\} \in q''.$$

Hence

$$(\mathfrak{B}, q'') \models (Q\, v_m)\, \psi[b_1, \ldots, b_n],$$

which proves our claim.

CLAIM 2. $q \cap S(A) = p$.

Proof. Obviously, $p \subseteq q \cap S(A)$.

Let $x \in q$ and $x \subseteq A$. Suppose $x \notin p$. Then we have $x \in q''$. Since q'' is minimal, we have

$$\{c \in B\colon (\mathfrak{B}, q'') \models \varphi[b_1, \ldots, c, \ldots, b_n]\} = x$$
$$\text{for some } \varphi \in L_Q \text{ and } b_1, \ldots, b_n \in B.$$

So we have $\{c \in A\colon (\mathfrak{B}, q'') \models \varphi[b_1, \ldots, c, \ldots, b_n]\} = x$. Since $(\mathfrak{B}, q'')$ is good with respect to $(\mathfrak{A}, p)$, x is finite. Then, by Proposition 2, $x \in p$. A contradiction.

From Claims 1 and 2 it follows that $(\mathfrak{B}, q)$ is a structure of cardinality $\varkappa$ which is an elementary 4-extension of $(\mathfrak{A}, p)$. This proves Theorem 2.5.

Note that, as follows from this theorem, we may omit the condition that p is minimal in Corollary 1 to Theorem 2.4.

The Upward Löwenheim-Skolem Theorem does not hold for $\prec_2$ and the Downward Löwenheim-Skolem Theorem does not hold for $\prec_2$, $\prec_3$ and $\prec_4$.

This is shown by the following counterexample.

Let L_Q have no individual constants and let R, F and G be three binary predicate letters. Let Σ_1 consist of the sentences

$$\forall v_0 \big(R(v_0, v_0)\big), \qquad \forall v_0 \forall v_1 \big(R(v_0, v_1) \leftrightarrow R(v_1, v_0)\big),$$
$$\forall v_0 \forall v_1 \forall v_2 \big(R(v_0, v_1) \wedge R(v_1, v_2) \rightarrow R(v_0, v_2)\big).$$

Let Σ_2 consist of the sentences

$$(Qv_0)(v_0 \neq v_0), \qquad \forall v_1 Q v_0 \big(R(v_0, v_1)\big),$$
$$\forall v_1 \ldots \forall v_n \Big(Qv_0 \varphi \rightarrow \exists v_{n+1} \forall v_0 \big(\varphi \leftrightarrow R(v_0, v_{n+1})\big)\Big) \quad \text{for all } \varphi(v_0, \ldots, v_n) \in L_Q.$$

Let Σ_3 consist of the sentences

$$\forall v_0 \forall v_1 \forall v_2 \big(F(v_0, v_1) \wedge F(v_0, v_2) \rightarrow v_1 = v_2\big),$$
$$\forall v_0 \exists v_1 \exists v_0 \big(F(v_1, v_2) \wedge R(v_0, v_2)\big), \qquad \forall v_0 \exists v_1 \big(F(v_0, v_1)\big),$$
$$\forall v_0 \forall v_1 \forall v_2 \big(F(v_0, v_2) \wedge F(v_1, v_2) \rightarrow v_0 = v_2\big),$$
$$\forall v_0 \forall v_1 \forall v_2 \forall v_3 \big(F(v_0, v_1) \wedge F(v_2, v_3) \wedge R(v_1, v_3) \rightarrow v_0 = v_2\big).$$

Let Σ_4 consist of the sentences

$$\forall v_0 \forall v_2 \exists v_1 \big(G(v_0, v_1) \wedge R(v_1, v_2)\big),$$
$$\forall v_0 \forall v_1 \forall v_2 \big(G(v_0, v_1) \wedge G(v_0, v_2) \rightarrow \neg R(v_1, v_2)\big), \qquad \forall v_1 \exists v_0 \big(G(v_0, v_1)\big),$$
$$\forall v_0 \forall v_1 \forall v_2 \forall v_3 \big(G(v_0, v_1) \wedge G(v_2, v_3) \wedge R(v_1, v_3) \wedge v_1 \neq v_3 \rightarrow v_0 \neq v_2\big).$$

Finally, let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$.

Then $(\mathfrak{A}, p) = \langle A, R^A, F^A, G^A, p\rangle$ with p minimal is a model of Σ iff the following conditions are satisfied:

(i) R^A is an equivalence relation;

(ii) p is exactly the set of equivalence classes;

(iii) F^A states a one-to-one mapping of A into itself such that $|E \cap \operatorname{rng} F| = 1$ for all equivalence classes E;

(iv) G^A is the union of all G^A_E's, where G^A_E is a bijection of A onto E for all equivalence classes E.

Σ has a model $\langle \omega - \{0\}, R^A, F^A, G^A, p\rangle$, where $R^A(n, m)$ iff $2^j | n$ iff $2^j | m$ for all $j \in \omega$. Let $\{E_n : n \in \omega - \{0\}\}$ be the set of equivalence classes. Then p is equal to this set.

$F^A(n, m)$ iff m is the minimum element of E_n. Let $f_n : E_n \to \omega - \{0\}$ be a bijection for all $n \in \omega - \{0\}$. Then $G^A(n, m)$ iff $f_j(m) = n$, where $m \in E_j$. By the Compactness Theorem Σ has models of each infinite cardinality.

Claim 1. *Let $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ be models of Σ such that $|A| < |B|$. Then it is not true that $(\mathfrak{A}, p) \prec_2 (\mathfrak{B}, q)$.*

Proof. q contains all equivalence classes of R^B. The cardinality of the set of equivalence classes is equal to $|B|$, and so there is a $y \in q$ such that $y \cap A = \emptyset$. Since $\emptyset \notin p$, it is not true that $(\mathfrak{A}, p) \prec_2 (\mathfrak{B}, q)$.

Claim 2. *Let $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ be models of Σ such that $|A| < |B|$ and q is minimal. Then neither $(\mathfrak{A}, p) \prec_3 (\mathfrak{B}, q)$ nor $(\mathfrak{A}, p) \prec_4 (\mathfrak{B}, q)$ holds.*

Proof. q is minimal, and so it contains only the equivalence classes of R^B, which are all of cardinality $|B|$. Because $|A| < |B|$, we have $p \nsubseteq q$.

3. Sentences preserved under the operation of forming extensions.

In this section we make an attempt to describe the set of sentences preserved under the operation of forming extensions. We distinguish four cases corresponding to the four definitions of a submodel and of an extension. Unfortunately, we did not succeed in giving a full description of these sets. However, in each case we shall give some characteristic examples. This has the following reason: when two or more definitions of a submodel and of an extension are given, one wants to see the differences between them by studying their properties. Especially, the differences between the sets of sentences preserved under the operation of forming extensions (or submodels) are a good indication of the differences between the given definitions. Moreover, in investigations of special relations between structures a description of the set of sentences preserved under the corresponding operation is a helpful tool (see, e.g., Węglorz [4]).

Definition. Let $\mathscr{E}$ be the *operation of forming extensions* of structures for L, i.e. $\mathfrak{B} \in \mathscr{E}(\mathfrak{A})$ if $\mathfrak{A} \subseteq \mathfrak{B}$.

Let $i \in \{1, 2, 3, 4\}$. $\mathscr{E}_i$ is the *operation of forming i-extensions* of structures for L_Q, i.e. $(\mathfrak{B}, q) \in \mathscr{E}_i(\mathfrak{A}, p)$ if $(\mathfrak{A}, p) \subseteq_i (\mathfrak{B}, q)$.

Let $\mathcal{O}$ be an operation defined on structures for L or L_Q. Then $\Delta(\mathcal{O})$ is the *set of structures in L or L_Q preserved under $\mathcal{O}$*, i.e. $\varphi \in \Delta(\mathcal{O})$ if $\mathscr{B} \models \varphi$ for all $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{B} \in \mathcal{O}(\mathscr{A})$ and $\mathscr{A} \models \varphi$ (here $\mathscr{A}$ and $\mathscr{B}$ are structures for L or L_Q).

We state the following obvious propositions without proof:

PROPOSITION 3. *Let $\mathcal{O}$ be an operation defined on structures for L or L_Q and let $H \subseteq \Delta(\mathcal{O})$. Then also $\bar{H} \subseteq \Delta(\mathcal{O})$, where $\bar{H}$ is the smallest set X such that $H \subseteq X$, and*

(i) $\varphi \wedge \psi$, $\varphi \vee \psi \in X$ *for all* $\varphi, \psi \in X$;
(ii) $\varphi \in X$ *whenever* φ *or* $\neg \varphi$ *is a tautology*;
(iii) $\psi \in X$ *whenever* $\varphi \in X$ *and* $\varphi \rightleftarrows \psi$ *is a tautology.*

PROPOSITION 4. $\Delta(\mathscr{E}) \subseteq \Delta(\mathscr{E}_i)$ *for each* $i \in \{1, 2, 3, 4\}$, $\Delta(\mathscr{E}_1) \subseteq \Delta(\mathscr{E}_i)$ *for each* $i \in \{2, 3, 4\}$, *and* $\Delta(\mathscr{E}_3) \subseteq \Delta(\mathscr{E}_4)$.

LEMMA (Łoś [3]). $\Delta(\mathscr{E}) = \{\psi \mid \psi$ *is a sentence in* L *such that* $\psi = \exists x_1 \ldots \exists x_n \varphi$, *where* φ *has no quantifier*$\}$.

Let $(\mathfrak{A}, p)$ be given. We shall investigate which properties of p can be expressed by a sentence of L_Q.

Definition. Let P be a property of structures for L_Q. P is a *first-order property* if there is a sentence $\sigma \in L_Q$ such that, for all structures $(\mathfrak{A}, p)$, $(\mathfrak{A}, p) \models \sigma$ iff $(\mathfrak{A}, p)$ has the property P.

Let $0 \leqslant n \leqslant m < \omega$ and $0 \leqslant k \leqslant l < \omega$ be such that if $k = l = 0$, then $0 \leqslant n \leqslant m \leqslant 1$.

$(\mathfrak{A}, p)$ has the property $P_{(n,m),(k,l)}$ if there are at least n and at most m elements x in p such that $k \leqslant |x| \leqslant l$.

$(\mathfrak{A}, p)$ has the property $U_{(n,m)}$ if $n \leqslant |A| \leqslant m$.

We can generalize this by writing, instead of n, m, k or l, the symbol ∞ which is interpreted as "infinite", e.g.,

$(\mathfrak{A}, p)$ has the property $P_{(n,\infty),(\infty,\infty)}$ if p has at least n infinite elements.

PROPOSITION 5. $U_{(n,m)}$ *is a first-order property iff* $0 \leqslant n \leqslant m < \omega$ *or* $0 \leqslant n < \omega$ *and* $m = \infty$.

PROPOSITION 6. $P_{(n,m),(k,l)}$ *is a first-order property iff one of the following conditions is satisfied*:

(i) $0 \leqslant k \leqslant l < \omega$ *and* $0 \leqslant n \leqslant m < \omega$;
(ii) $0 \leqslant k \leqslant l < \omega$ *and* $0 \leqslant n < \omega$ *and* $m = \infty$;
(iii) $n = 0$ *and* $m = \infty$.

Proof. If $n = 0$ and $m = \infty$, then there is nothing to prove, since every structure for L_Q has the property $P_{(0,\infty),(k,l)}$.

If $0 \leqslant n \leqslant m < \omega$ and $0 \leqslant k \leqslant l < \omega$, then $P_{(n,m),(k,l)}$ is a positive Boolean combination of $P_{(i,i),(j,j)}$'s, where $n \leqslant i \leqslant m$ and $k \leqslant j \leqslant l$. It is easy to see that $P_{(i,i),(j,j)}$ for all $i, j < \omega$ is a first-order property.

Let $0 \leqslant n < \omega$, $m = \infty$ and $0 \leqslant k \leqslant l < \omega$. Let $q_{k,l}(y_1, \ldots, y_l)$ or, shortly, $q_{k,l}(y_i)$ be a formula expressing $k \leqslant |\{y_1, \ldots, y_l\}| \leqslant l$, and let $\sigma_l(y_1, \ldots, y_l, z_1, \ldots, z_l)$ or, shortly, $\sigma_l(y_i, z_i)$ be a formula expressing $\{y_1, \ldots, y_l\} \neq \{z_1, \ldots, z_l\}$. Then $P_{(n,\infty),(k,l)}$, where $n \geqslant 1$, is expressed by the following sentence:

$$\exists y_{11} \ldots \exists y_{1l} \ldots \exists y_{n1} \ldots \exists y_{nl} \big(q_{k,l}(y_{1i}) \wedge \ldots \wedge q_{k,l}(y_{ni}) \wedge \sigma_l(y_{1i}, y_{2i}) \wedge \ldots \wedge$$
$$\wedge\, \sigma_l(y_{1i}, y_{ni}) \wedge \ldots \wedge \sigma_l(y_{n-1,i}, y_{ni}) \wedge$$
$$\wedge\, (Qz)(z = y_{11} \vee \ldots \vee z = y_{1l}) \wedge \ldots \wedge (Qz)(z = y_{n1} \vee \ldots \vee z = y_{nl})\big).$$

Conversely, let $P_{(n,m),(k,l)}$ be given. We consider three cases.

(a) $n = m = \infty$. Then $P_{(n,m),(k,l)}$ is not a first-order property, as is easy to see with the use of the ultraproduct construction.

(b) (i) $1 \leqslant n \leqslant m < \omega$ and $l = \infty$.

(ii) $1 \leqslant n < \omega$ and $m = l = \infty$.

In each of these cases the following counterexample shows that $P_{(n,m),(k,l)}$ is not a first-order property. Indeed, for $i \in \omega$, let p_i be the i-th prime number and $p_i Z = \{n \in \boldsymbol{Z}\colon p_i | n\}$. In the same way as at the end of Section 6 we prove that, for all $\varphi \in L_Q$ and $a_1, \ldots, a_n \in \boldsymbol{Z}$,

$$\langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant n+1\}\rangle \models \varphi[a_1, \ldots, a_n]$$
$$\text{iff } \langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant n\}\rangle \models \varphi[a_1, \ldots, a_n].$$

$\langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant n+1\}\rangle$ has the property $P_{(n,m),(k,l)}$, but $\langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant n\}\rangle$ does not.

(c) $n = 0$, $m < \omega$ and $l = \infty$. Also in this case we have a counterexample which shows that $P_{(n,m),(k,l)}$ is not a first-order property. Namely, as in case (b), $\langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant m+1\}\rangle$ has the property $P_{(n,m),(k,l)}$, while $\langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant m+2\}\rangle$ does not. But also, for all $\varphi \in L_Q$ and $b_1, \ldots, b_n \in \boldsymbol{Z}$, we have

$$\langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant m+1\}\rangle \models \varphi[b_1, \ldots, b_n]$$
$$\text{iff } \langle \boldsymbol{Z}, \{p_i \boldsymbol{Z}\colon 2 \leqslant i \leqslant m+2\}\rangle \models \varphi[b_1, \ldots, b_n].$$

This completes the proof of the lemma.

Let $\pi_{(n,m),(k,l)}$ be a sentence of L_Q expressing $P_{(n,m),(k,l)}$ if it is a first-order property. Let $\nu_{(n,m)}$ be a sentence of L expressing $U_{(n,m)}$ if it is a first-order property. It is easy to see that we have the following

PROPOSITION 7. (i) $\Delta(\mathscr{E}) \cup \{\nu_{(n+1,\infty)} \vee \pi_{(m,\infty),(0n)}$, *for all* $n, m < \omega\} \subseteq \Delta(\mathscr{E}_1)$.

(ii) $\Delta(\mathscr{E}_1) \cup \{\psi \mid \psi = \exists x_1 \ldots \exists x_n \neg Qy\varphi$, *where* φ *is open*$\} \subseteq \Delta(\mathscr{E}_2)$.

(iii) $\Delta(\mathscr{E}_1) \cup \{\pi_{(n,\infty),(k,l)}, \neg \pi_{(0,m),(k,l)}, \pi_{(1,1),(0,0)}$ *for* $n, m, k, l < \omega\} \subseteq \Delta(\mathscr{E}_3)$.

(iv) $\Delta(\mathscr{E}_3) \cup \{\neg \pi_{(1,1),(0,0)}\} \subseteq \Delta(\mathscr{E}_4)$.

From this it follows also that all inclusions in Proposition 4 are strict. The above-given propositions show what, given a structure $(\mathfrak{A}, p)$, can be said about p. The author did not succeed in giving a full description, which is necessary to prove some preservation theorems.

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