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ON STARSHAPEDNESS OF THE UNION OF CLOSED SETS IN $\boldsymbol{R}^n$

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In this note we give a theorem on starshapedness of the union of a finite family of closed sets in $\boldsymbol{R}^n$. This theorem is equivalent to Helly's theorem for a finite family of convex sets.

1. Basic definitions and notation. Let S be a subset of $\boldsymbol{R}^n$. For points x and y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ is contained in S. The *star of x with respect to S* is the set $\mathrm{st}(x, S)$ of all points of S which see x via S. We say that a set S is *starshaped* if it contains a point q which sees every point of S. The set of all such points q is called the *kernel* of S and is denoted by $\ker(S)$. Obviously, $\ker(S)$ can be alternatively defined as the intersection $\bigcap\{\mathrm{st}(x, S)\colon x \in S\}$.

Throughout the note, $\mathrm{conv}(S)$, $\dim(S)$, $\mathrm{card}(S)$ will denote the convex hull, dimension, cardinality, respectively, of the set S. Finally, let $B(0, M) = \{x\colon \|x\| \leqslant M\}$ and let $\langle n\rangle$ denote the set of natural numbers $\{1, 2, \ldots, n\}$.

2. The result. A well-known theorem of Krasnosel'skiĭ (see [7] and [10]) states that if S is a compact set in $\boldsymbol{R}^n$, then S is starshaped if and only if any $n+1$ points of S see a common point via S. This result was expanded by Valentine [10]. The Valentine theorem can be formulated as follows:

THEOREM 1. *If S is a closed set in $\boldsymbol{R}^n$ and C is a compact subset of S such that any $n+1$ points in S can see a common point in C via S, then S is starshaped.*

As a consequence of this theorem we obtain the following result:

THEOREM 2. *Let $\mathscr{F}$ be a finite family of closed sets in $\boldsymbol{R}^n$. If any $n+1$ members of $\mathscr{F}$ have a starshaped union, then* $\ker(\bigcup \mathscr{F}) \neq \emptyset$.

Proof. Let $\mathscr{F}$ be an m-membered ($m \geqslant n+1$) family of closed sets in $\boldsymbol{R}^n$. Consider all subfamilies $\mathscr{G}_1, \ldots, \mathscr{G}_s$ of $\mathscr{F}$,

$$s = \binom{m}{n+1},$$

having $n+1$ members. By assumption we have

$$\ker(\bigcup \mathscr{G}_i) \neq \emptyset \quad \text{for } i \in \langle s \rangle.$$

Obviously, the sets $\bigcup \mathscr{G}_i$, $i \in \langle s \rangle$, are closed. This implies that the sets $\ker(\bigcup \mathscr{G}_i)$ are also closed (see the Lemma in [6]).

Now we take a ball $B(0, M)$ such that

$$B(0, M) \cap \ker(\bigcup \mathscr{G}_i) \neq \emptyset \quad \text{for } i \in \langle s \rangle,$$

and consider the following sets:

$$S = \bigcup \mathscr{F},$$
$$C = \bigcup \{B(0, M) \cap \ker(\bigcup \mathscr{G}_i) \colon i \in \langle s \rangle\}.$$

It is easy to verify that these sets satisfy the assumptions of Theorem 1. Thus $\ker(\bigcup \mathscr{F}) \neq \emptyset$ and the proof is completed.

Following Peterson [8] we call a set S *finitely starlike* if every finite subset of S can see a common point via S.

By Krasnosel'skiĭ's theorem and Theorem 2 we obtain immediately the following

COROLLARY. *Let $\mathscr{F}$ be a family of closed sets in $\mathbf{R}^n$ such that any $n+1$ members of $\mathscr{F}$ have a starshaped union. Then*

(i) *if $\bigcup \mathscr{F}$ is compact, then* $\ker(\bigcup \mathscr{F}) \neq \emptyset$;

(ii) *if $\mathscr{F}$ is finite, then* $\ker(\bigcup \mathscr{F}) \neq \emptyset$;

(iii) *if $\mathscr{F}$ is infinite, then $\bigcup \mathscr{F}$ is a finitely starlike set.*

The notion of dimension of the set $\ker(S)$ was investigated by many authors (see, e.g., [2]–[4], [9]). The study of $\dim(\ker(S))$ has been stimulated by Problem 1.1 in [10].

In the subsequent theorem we give conditions that will guarantee that the kernel of the union of all members of $\mathscr{F}$ is at least k-dimensional. That theorem contains Theorem 2 as the case $k = 0$.

First we recall the well-known equality

$$\bigcap \{\mathrm{st}(x, S) \colon x \in S\} = \bigcap \{\mathrm{conv}(\mathrm{st}(x, S)) \colon x \in S\}, \tag{1}$$

which is valid for any closed set $S \subset \mathbf{R}^n$, and define

$$g(n, k) = \begin{cases} n+1 & \text{if } k = 0, \\ \max\{n+1, 2n-2k+2\} & \text{if } 1 \leqslant k \leqslant n. \end{cases}$$

Moreover, we quote here the following theorem of Katchalski [5] which we will need:

THEOREM 3. *Let $\mathscr{F}$ be a finite collection of convex sets in $\mathbf{R}^n$ such that* *each subfamily of $g(n, k)$ members has intersection of dimension at least k. Then* $\dim(\bigcap \mathscr{F}) \geqslant k$.

Let us remark that the case $k = 0$ in Theorem 3 is Helly's theorem.

THEOREM 4. *Let $\mathscr{K}$ be a finite collection of closed sets in $\boldsymbol{R}^n$ such that the union of each subfamily of $g(n, k)$ members has at least k-dimensional kernel. Then* $\dim(\ker(\bigcup \mathscr{K})) \geqslant k$.

Proof. The proof proceeds by induction on $s = \operatorname{card} \mathscr{K}$. The theorem is obviously true for $s = g(n, k)$. Suppose the result holds for $s = s_0 \geqslant g(n, k)$, and consider the case $s = s_0 + 1$. Our induction assumption implies

$$\dim(\ker(\bigcup \mathscr{K}_{\hat{i}})) \geqslant k \quad \text{for } i \in \langle s_0 + 1 \rangle, \tag{2}$$

where $\mathscr{K}_{\hat{i}}$ (here and further on) denotes the set $\mathscr{K} \setminus \{S_i\}$.

We always have

$$\begin{aligned}
\ker(\bigcup \mathscr{K}_{\hat{i}}) &= \bigcap \{\operatorname{st}(x, \bigcup \mathscr{K}_{\hat{i}}) \colon x \in \bigcup \mathscr{K}_{\hat{i}}\} \\
&= \bigcap \{\bigcap \{\operatorname{st}(x, \bigcup \mathscr{K}_{\hat{i}}) \colon x \in S_j\} \colon j \in \langle s_0 + 1 \rangle \setminus \{i\}\} \\
&\subset \bigcap \{\bigcap \{\operatorname{st}(x, \bigcup \mathscr{K}) \colon x \in S_j\} \colon j \in \langle s_0 + 1 \rangle \setminus \{i\}\} \\
&\subset \bigcap \{\bigcap \{\operatorname{conv}(\operatorname{st}(x, \bigcup \mathscr{K})) \colon x \in S_j\} \colon j \in \langle s_0 + 1 \rangle \setminus \{i\}\}.
\end{aligned} \tag{3}$$

By Q_j we denote the following convex sets:

$$Q_j = \bigcap \{\operatorname{conv}(\operatorname{st}(x, \bigcup \mathscr{K})) \colon x \in S_j\}, \quad j \in \langle s_0 + 1 \rangle.$$

From (2) and (3) it follows that the intersection of any s_0 sets Q_j is at least k-dimensional. Katchalski's theorem now implies that

$$\dim(\bigcap \{Q_j \colon j \in \langle s_0 + 1 \rangle\}) \geqslant k. \tag{4}$$

Obviously, the set $\bigcup \mathscr{K}$ satisfies (1), and therefore we have

$$\begin{aligned}
\ker(\bigcup \mathscr{K}) &= \bigcap \{\operatorname{conv}(\operatorname{st}(x, \bigcup \mathscr{K})) \colon x \in \bigcup \mathscr{K}\} \\
&= \bigcap \{\bigcap \{\operatorname{conv}(\operatorname{st}(x, \bigcup \mathscr{K})) \colon x \in S_j\} \colon j \in \langle s_0 + 1 \rangle\} \\
&= \bigcap \{Q_j \colon j \in \langle s_0 + 1 \rangle\}.
\end{aligned}$$

These equalities and (4) imply $\dim(\ker(\bigcup \mathscr{K})) \geqslant k$. This completes the proof.

3. Equivalence of Helly's theorem and Theorem 2. Krasnosel'skiĭ's theorem is proved by applying Helly's theorem to an infinite family of convex sets (in this case the compactness of the sets must be additionally assumed). Borwein [1] has shown that Helly's theorem (for a finite family) can be deduced from Krasnosel'skiĭ's theorem. In this sense both theorems are equivalent. Our Theorem 2 is obtained from Helly's theorem for a finite collection (cf. the proof of Theorem 4). The following natural question arises: Can Helly's theorem be deduced from Theorem 2? The answer is positive, and the proof of this fact goes just as in [1], but starshapedness of the set

$$S = \bigcup \{E_j \cap B(0, M)\colon\ n_1 \leqslant j \leqslant n_{N+1}\}$$

is obtained from Theorem 2.

Let us note that using arguments similar to the ones above we can show the equivalence of Katchalski's theorem and our Theorem 4.

4. Examples. Finally, we give three examples demonstrating that the closedness and finiteness conditions in Theorem 2 cannot be dropped and that the number $n+1$ in Theorem 2 is the best possible. Similar examples showing the importance of all the assumptions in Theorem 4 may also be given.

EXAMPLE 1. Consider a family $\mathscr{F}$ of all $(n-1)$-dimensional closed sides of an n-dimensional simplex. Here $\ker(\bigcup \mathscr{F}_i) \neq \emptyset$ for $i \in \langle n+1 \rangle$, but $\ker(\bigcup \mathscr{F}) = \emptyset$. This shows that the positive integer $n+1$ is the best possible.

EXAMPLE 2. Let $\mathscr{K}$ be a family of $n+2$ closed convex cones $C_1, \ldots, C_{n+2}$, all with the apex at the origin, such that $\bigcup \mathscr{K}$ covers $\boldsymbol{R}^n$ and $\bigcup \mathscr{K}_i$, $i \in \langle n+2 \rangle$, does not cover $\boldsymbol{R}^n$.

Now we consider a family $\mathscr{F}$ of sets S_i which take the form $S_i = C_i \setminus \{0\}$.

The union of any $n+1$ members of $\mathscr{F}$ is a starshaped set because, as is easy to verify,

$$-S_i \subset \ker(\bigcup \mathscr{F}_i) \quad \text{for } i \in \langle n+2 \rangle,$$

where $-S_i = \{-x\colon\ x \in S_i\}$. In this case we have

$$\bigcup \mathscr{F} = \boldsymbol{R}^n \setminus \{0\},$$

and thus $\ker(\bigcup \mathscr{F}) = \emptyset$. This shows that the closedness assumption in Theorem 2 cannot be omitted.

EXAMPLE 3. Let $\mathscr{F}$ be an infinite collection of closed sets $S_1, S_2, \ldots,$ where

$$S_i = \{(x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n\colon\ 0 \leqslant x_1 \leqslant i,\ x_2 \leqslant E(x_1+1) - x_1\}$$

and $E(x)$ denotes the greatest integer not exceeding x.

It is easy to check that the union of any $n+1$ members of $\mathscr{F}$ is a starshaped set, but $\ker(\bigcup \mathscr{F}) = \emptyset$. This shows that the finiteness condition in Theorem 2 is essential.

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