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CHARACTERIZATIONS OF METRIC COMPLETENESS

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In this paper, we give some necessary and sufficient conditions for a metric space (X, d) to be complete. Such characterizations of metric completeness are given mainly by results relevant to Caristi's fixed point theorem. Works of Cantor, Kuratowski, Ekeland, Caristi, Kirk, Wong, Weston, Ćirić, Hu, Reich, Subrahmanyam, and others are combined.

Kuratowski [18] first noticed that the Cantor intersection theorem characterizes the metric completeness. Hu [12] showed that a metric space is complete if and only if any Banach contraction on closed subsets thereof has a fixed point. On the other hand, Kirk [15] showed that Caristi's theorem characterizes the metric completeness. Later, motivated by Wong's proof [27] of Caristi's theorem, Weston [26] showed that a metric space X is complete if and only if X satisfies a condition of Ekeland [10], [11], that is, for each lower semicontinuous function $h: X \to (-\infty, \infty)$ bounded from below on X, there is a point p in X such that $h(p)-h(x) < d(p, x)$ for every point x in X. Reich [23] and Subrahmanyam [25] also obtained characterizations of the metric completeness using Kannan's result [13] similar to the Banach contraction principle, which is known to be a consequence of Caristi's theorem. On the other hand, Kolodner [17] and Boyd and Wong [2] noticed that the Banach contraction principle follows from the Cantor intersection theorem.

Now we combine those results and state our characterizations of the metric completeness. Let ω denote the set of nonnegative integers and $^-$ the closure operation.

THEOREM. *For a metric space (X, d), the following statements are equivalent*:

(i) *X is complete.*

(ii) *For every sequence $\{\alpha_n\}_{n\in\omega}$ of positive numbers converging to* 0 *and every sequence $\{F_n\}_{n\in\omega}$ of nonempty closed subsets of X such that $F_{n+1} \subset F_n$, $n\in\omega$, and each F_n is a union of finite number of subsets of diameter less than α_n, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

(iii) *For every sequence* $\{F_n\}_{n\in\omega}$ *of nonempty closed subsets of* X *such that* $F_{n+1} \subset F_n$, $n \in \omega$, *and the sequence* $\{\operatorname{diam} F_n\}_{n\in\omega}$ *converges to* 0, *we have* $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

(iv) *Every lower semicontinuous function* h: $X \to (-\infty, \infty)$ *which is bounded from below has a d-point* p *in* X, *that is,*

$$h(p)-h(x) < d(p, x)$$

for every point x *in* X, $x \neq p$.

(v) *For every selfmap* f *of* X *with a lower semicontinuous function* V: $X \to (-\infty, \infty)$ *which is bounded from below and such that, for each* x *in* X *with* $x \neq fx$, *there exists* y *in* $X-\{x\}$ *satisfying*

$$d(x, y) \leqslant V(x)-V(y),$$

f *has a fixed point.*

(vi) *For every selfmap* f *of* X *such that there exists a lower semicontinuous function* φ: $X \to (-\infty, \infty)$ *which is bounded from below and satisfies*

$$d(x, fx) \leqslant \varphi(x)-\varphi(fx)$$

for each x *in* X, f *has a fixed point.*

(vii) *For every selfmap* f *of* X *such that there exist a* $u \in X$ *and an* $\alpha \in [0, 1)$ *satisfying*

$$d(fx, f^2x) \leqslant \alpha d(x, fx)$$

for each x *in* $\overline{\{f^n u\}_{n\in\omega}}$ *and* f *is continuous on* $\overline{\{f^n u\}}$, f *has a fixed point in* $\overline{\{f^n u\}}$.

(viii) *For every selfmap* f *of* X *such that there exist a* $u \in X$ *and an* $\alpha \in [0, 1)$ *satisfying*

$$d(fx, fy) \leqslant \alpha \max\{d(x, y),\ d(x, fx),\ d(y, fy),\ [d(x, fy)+d(y, fx)]/2\}$$

for all $x, y \in \overline{\{f^n u\}}$, f *has a (unique) fixed point in* $\overline{\{f^n u\}}$.

Proof. (i)$\Rightarrow$(ii) is given in [18], [19], and (ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(iv). We order X by defining $x \leqslant y$ iff $d(x, y) \leqslant h(x)-h(y)$. For each $x \in X$, let $X(x) = \{y \in X \mid x \leqslant y\}$. We construct an increasing sequence $\{x_n\}$ as follows: Choose $x_0 \in X$ arbitrarily, and if $x_0, \ldots, x_n$ are given, then choose $x_{n+1} \in X(x_n)$ with $h(x_{n+1}) < \inf h(X(x_n)) + 1/n$. Thus $x_n \leqslant x_{n+1}$ and for each $x \in X(x_{n+1}) \subset X(x_n)$ we have

$$h(x_{n+1})-1/n < \inf h(X(x_n)) \leqslant h(x)$$

and

$$d(x, x_{n+1}) \leqslant h(x_{n+1})-h(x).$$

Hence $\operatorname{diam} X(x_{n+1}) \leqslant 2/n$. By (iii), the intersection of the decreasing sequence of nonempty closed sets $\{X(x_n)\}$ contains exactly one point, say p. Since $p \leqslant x$ implies $x \in X(x_n)$ for all n, p is a maximal point and also a d-point of X.

(iv) $\Rightarrow$ (v). By (iv), V has a d-point $p \in X$. Suppose $p \neq fp$. Then there exists $y \in X \setminus \{p\}$ satisfying $d(p, y) \leqslant V(p) - V(y)$ by assumption. However, we have $V(p) - V(y) < d(p, y)$, a contradiction. Hence p is a fixed point of f.

(v) $\Rightarrow$ (vi). Put $y = fx$ in (v).

(vi) $\Rightarrow$ (vii). Since

$$d(x, fx) - \alpha d(x, fx) \leqslant d(x, fx) - d(fx, f^2 x),$$

putting $\varphi(x) = (1-\alpha)^{-1} d(x, fx)$ we have $d(x, fx) \leqslant \varphi(x) - \varphi(fx)$ for each $x \in \overline{\{f^n u\}}$, and φ: $\overline{\{f^n u\}} \to [0, \infty)$ is continuous.

(vii) $\Rightarrow$ (viii). Putting $y = fx$, we have

$$\begin{aligned} d(fx, f^2 x) &\leqslant \alpha \max\{d(x, fx),\ d(fx, f^2 x),\ d(x, f^2 x)/2\} \\ &= \alpha \max\{d(x, fx),\ [d(x, fx) + d(fx, f^2 x)]/2\} \\ &\leqslant \alpha d(x, fx). \end{aligned}$$

Suppose there exists a subsequence $\{f^{n_i} u\}$ converging to some $p \in \overline{\{f^n u\}}$. Since $\{f^n u\}$ is Cauchy, $f^n u \to p$ and

$$\begin{aligned} & d(fp, f^{n+1} u) \\ & \leqslant \alpha \max\{d(p, f^n u),\ d(p, fp),\ d(f^n u, f^{n+1} u),\ [d(p, f^{n+1} u) + d(f^n u, fp)]/2\} \end{aligned}$$

implies

$$d(fp, p) \leqslant \alpha d(fp, p).$$

This shows that $fp = p$ and $f | \overline{\{f^n u\}}$ is continuous. Therefore, f satisfies the hypothesis of (vii).

(viii) $\Rightarrow$ (i). Suppose that X contains a nonconvergent Cauchy sequence $\{x_n\}_{n \in \omega}$. We may assume that $\{x_n\}$ consists of distinct terms. Take any $x \in X$; then

$$l(x) := \inf\{d(x, x_n) \mid x_n \neq x,\ n \in \omega\} > 0$$

because $\{x_n\}$ has no cluster point. Choose any $\alpha \in (0, 1)$. We define σ: $\omega \to \omega$ inductively as follows: $\sigma(0) := 0$; and if $n \geqslant 1$ and $\sigma(n-1)$ is defined, let $\sigma(n)$ be an integer greater than $\sigma(n-1)$ such that $d(x_i, x_j) \leqslant \alpha l(x_{\sigma(n-1)})$ for all integers $i, j \geqslant \sigma(n)$. Then $\{x_{\sigma(n)}\}_{n \in \omega}$ is a subsequence of distinct terms and does not converge. The set $\{x_{\sigma(n)}\}$ is closed and the map f: $X \to X$ defined by

$$f(x) = x_0 \text{ if } x \notin \{x_{\sigma(n)}\} \quad \text{and} \quad f(x_{\sigma(n)}) = x_{\sigma(n+1)}$$

is a Banach contraction on $\overline{\{f^n x_0\}}$; hence f satisfies the hypothesis of (viii) on $\overline{\{f^n x_0\}}$. However, f does not have a fixed point.

This completes our proof.

Remarks. (1) Kuratowski [18] obtained (i) $\Rightarrow$ (ii) as a generalization of the Cantor intersection theorem (i) $\Rightarrow$ (iii) (see [19]). He also noticed that (i) $\Leftrightarrow$ (iii).

(2) (i) $\Rightarrow$ (iv) was actually due to Ekeland [10], [11]. Weston proved (i) $\Leftrightarrow$ (iv). Our proof of (iii) $\Rightarrow$ (iv) is based on the proof of (i) $\Rightarrow$ (vi) of Penot [22].

(3) Caristi's fixed point theorem (i) $\Rightarrow$ (vi) with $\varphi\colon X \to [0, \infty)$ was given in [7]. It is actually equivalent to (i) $\Rightarrow$ (iv) announced in 1972 by Ekeland [10], whose result is an abstraction of a lemma due to Bishop and Phelps [1]. Various proofs of Caristi's theorem were given by Brønsted [4], [5], Browder [6], Kasahara [14], Kirk [15], Pasicki [21], Penot [22], Siegel [24], and Wong [27]. Condition (vi) was due to Brønsted [5].

(4) Kirk [15] showed (i) $\Leftrightarrow$ (vi). Wong [27] claimed that (i) $\Rightarrow$ (vi) implies (i) $\Rightarrow$ (v). A proof of (v) $\Rightarrow$ (iv) was also given by Wong [28]. Brézis and Browder [3] showed that (i) $\Rightarrow$ (vi) is equivalent to (i) $\Rightarrow$ (iv).

(5) Kirk and Caristi [16] noted that (i) $\Rightarrow$ (vi) implies the Banach contraction principle. Weston [26] noted that, by putting

$$\varphi(x) = (1-2\alpha)^{-1}(1-\alpha)\, d(x, fx),$$

the contractive type condition

$$d(fx, fy) \leqslant \alpha\, \{d(x, fy) + d(y, fx)\}, \qquad \alpha < 1/2,$$

implies the hypothesis of (vi).

(6) In the proof of (vi) $\Rightarrow$ (vii), f and φ are continuous on $\overline{\{f^n u\}}$. Browder [6] observed that, in (i) $\Rightarrow$ (vi), if f is continuous, then $\lim f^n x$ exists for all $x \in X$ and it is fixed under f.

(7) A variant of (i) $\Rightarrow$ (viii) was first given by Ćirić [8], and later extended by a number of authors. Pal and Maiti [20] considered an extended form of (viii), which is a particular case of (vii).

(8) The basic idea of the proof of (viii) $\Rightarrow$ (i) is due to Hu [12]. Reich [23] used Hu's idea with respect to Kannan's contractive condition [13]:

$$d(fx, fy) \leqslant \alpha\, \{d(x, fx) + d(y, fy)\}, \qquad \alpha < 1/2.$$

Similar results are given also by Subrahmanyam [25].

(9) Kolodner [17] and Boyd and Wong [2] noticed that the Cantor intersection theorem implies the Banach contraction principle. However, using an example of Connell [9], Subrahmanyam [25] noticed that the

Banach principle does not characterize the metric completeness, that is, we cannot claim that a metric space is complete if any contraction on it has a fixed point.

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