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A NOTE ON AUTOMORPHISMS AND PARTITIONS

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A family $\mathcal{F} \subseteq \mathcal{P}(T)$ is a *field* of subsets of a set T if $\mathcal{F}$ is closed under finite unions of its elements, complementations and contains all one-element sets $\{x\}$ for $x \in T$. A family $\mathcal{I} \subseteq \mathcal{P}(T)$ is an *ideal* of subsets of T if $\mathcal{I}$ is closed under finite unions, any subset of a set from $\mathcal{I}$ is a set from $\mathcal{I}$ and $\bigcup \mathcal{I} = T$. A field $\mathcal{F}$ (an ideal $\mathcal{I}$) is *proper* if $\mathcal{F} \neq \mathcal{P}(T)$ (if $T \in \mathcal{I}$).

With a given field $\mathcal{F}$ of subsets of T we can correlate an ideal $\mathcal{I}_{\mathcal{F}}$ of subsets of T by:

$$X \in \mathcal{I}_{\mathcal{F}} \quad \text{iff} \quad \mathcal{P}(X) \subseteq \mathcal{F}.$$

A permutation f of T is an *automorphism* of $\mathcal{F}$ if for each $X \in \mathcal{P}(T)$ we have: $X \in \mathcal{F}$ iff $f[X] \in \mathcal{F}$. Notice that if f is an automorphism of $\mathcal{F}$, then $f[\mathcal{I}_{\mathcal{F}}] = \mathcal{I}_{\mathcal{F}}$. An automorphism f of $\mathcal{F}$ is *trivial* iff

$$\{x \in T\colon f(x) \neq x\} \in \mathcal{I}_{\mathcal{F}}.$$

The aim of this note* is to prove the following theorem:

THEOREM. *If a field $\mathcal{F}$ of subsets of T has a non-trivial automorphism, then there is a partition $\mathcal{U}$ of T into at most two-element sets without any selector in $\mathcal{F}$.*

This theorem is a kind of a counterpart of the well-known Vitali construction of a non-measurable set. A similar result for σ-fields has been obtained by Cichoń and Porada in [1], but their proof depends deeply on countable additivity of a given field. In the proof presented here, we shall use some ideas occurring in our paper [3].

DEFINITION. We say that a field $\mathcal{F}$ of subsets of T has the *property* $S(2)$ (the *property CP*) iff every partition $\mathcal{U}$ of T into at least (at most, respectively) two-element sets has a selector in $\mathcal{F}$.

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* The main idea of the proof was found while the author was a Postdoctoral Fellow at the Waikato University, Hamilton, New Zealand, during the spring of 1980.

Proposition 1. *For any field $\mathscr{F}$ of subsets of T, the following three conditions are equivalent:*

(a) *$\mathscr{F}$ has the property $S(2)$;*

(b) *$\mathscr{F}$ has the property CP;*

(c) *for every family $\mathscr{V}$ of pairwise disjoint at least two-element subsets of T, there is a selector of $\mathscr{V}$ in $\mathscr{F}$.*

Proof. (c) $\Rightarrow$ (b) Let $\mathscr{U}$ be a partition of T into at most two-element sets. Let $\mathscr{V} = \{X \in \mathscr{U}: |X| = 2\}$. By (c), there is a selector $S \in \mathscr{F}$ of $\mathscr{V}$. But then $T-S$ is a selector of $\mathscr{U}$.

(b) $\Rightarrow$ (a) Let $\mathscr{U} = \{U_\alpha: \alpha < \delta\}$ be a partition of T into at least two-element sets. For each $\alpha < \delta$, fix a two-element subset V_α of U_α and put $A = \bigcup_{\alpha<\delta} V_\alpha$. Let

$$\mathscr{V} = \{V_\alpha: \alpha < \delta\} \cup \{\{x\}: x \in T-A\}.$$

Then $\mathscr{V}$ is a partition of T into at most two-element subsets of T. By (b), there is a selector $S \in \mathscr{F}$ of $\mathscr{V}$. But then $T-S \in \mathscr{F}$ is a selector of $\mathscr{U}$.

(a) $\Rightarrow$ (c) Let $\mathscr{V} = \{V_\alpha: \alpha < \delta\}$ be a family of pairwise disjoint at least two-element subsets of T. Let $A = \bigcup_{\alpha<\delta} V_\alpha$. Consider the partition

$$\mathscr{U} = \{V_0 \cup (T-A)\} \cup \{V_{1+\alpha}: \alpha < \delta\}$$

of T. Each member of $\mathscr{U}$ has at least two elements. By (a), there is a selector $S \in \mathscr{F}$ of $\mathscr{U}$. Pick $x_0 \in V_0$. Now, it is easy to see that $(S \cap A - V_0) \cup \{x_0\} \in \mathscr{F}$ is a selector of $\mathscr{V}$.

Definition. Two subsets X and Y of T are *$\mathscr{F}$-separable* if there is $A \in \mathscr{F}$ such that $X \subseteq A$ and $A \cap Y = 0$.

Proposition 2. *Let X and Y be $\mathscr{F}$-separable. Then $X \cup Y \in \mathscr{F}$ iff both X and Y are in $\mathscr{F}$.*

The proof is obvious.

Lemma 1. *Suppose X and Y are $\mathscr{F}$-separable subsets of T and f is an automorphism of $\mathscr{F}$ such that $f[X] = Y$. If $\mathscr{F} \in S(2)$, then $X \in \mathscr{F}$.*

Proof. Consider a partition $\mathscr{U}$ of $X \cup Y$ into sets of the form $\{x, f(x)\}$, where $x \in X$. By Proposition 1, (a) $\Rightarrow$ (c), there is a selector S of $\mathscr{U}$ in $\mathscr{F}$. Now, by $\mathscr{F}$-separability of X and Y, we see that $X \cap S \in \mathscr{F}$ and $Y \cap S \in \mathscr{F}$. But then

$$X = (X \cap S) \cup f^{-1}[Y \cap S] \in \mathscr{F}.$$

Thus $X \in \mathscr{F}$.

Definition. With any $X \subseteq T$ we correlate an ideal $\mathscr{I}_X$ on X by: $Y \in \mathscr{I}_X$ iff there is $A \in \mathscr{F}$ such that $Y \subseteq A \subseteq X$.

Proposition 3. (i) *$\mathscr{I}_X$ is a proper ideal on X iff $X \notin \mathscr{F}$.*

(ii) *If f is an automorphism of $\mathscr{F}$, then $f[\mathscr{I}_X] = \mathscr{I}_{f[X]}$.*

(iii) *If $Z \subseteq X$ and $Z \notin \mathscr{I}_X$, then $Z \notin \mathscr{F}$.*

(iv) *If X and Y are $\mathscr{F}$-separable sets, then $\mathscr{I}_{X \cup Y}$ is the smallest ideal on $X \cup Y$ that contains both $\mathscr{I}_X$ and $\mathscr{I}_Y$.*

Proof. (i), (ii) and (iii) are obvious, so we prove (iv). First, notice that both $\mathscr{I}_X$ and $\mathscr{I}_Y$ are always contained in $\mathscr{I}_{X \cup Y}$. Thus take any $Z \in \mathscr{I}_{X \cup Y}$; i.e., there is $A \in \mathscr{F}$ such that $Z \subseteq A \subseteq X \cup Y$. Since X and Y are $\mathscr{F}$-separable, there is $B \in \mathscr{F}$ such that $X \subseteq B$ and $Y \cap B = 0$. But then

$$Z \cap B \subseteq A \cap B \subseteq X \quad \text{and} \quad Z - B \subseteq A - B \subseteq Y.$$

Consequently, $Z \cap B \in \mathscr{I}_X$ and $Z - B \in \mathscr{I}_Y$, i.e., Z belongs to the smallest ideal containing both $\mathscr{I}_X$ and $\mathscr{I}_Y$.

LEMMA 2. *Let f be an automorphism of a field $\mathscr{F}$ and let*

$$C = \{x \in T \colon f(x) \neq x \text{ and } f^2(x) \neq x\}.$$

If $\mathscr{F} \in S(2)$, then $C \in \mathscr{I}_{\mathscr{F}}$.

Proof. Notice that $f[C] = C$. To get a contradiction suppose $C \notin \mathscr{I}_{\mathscr{F}}$. Thus choose $Z \subseteq C$ such that $Z \notin \mathscr{F}$. Let $\mathscr{M}$ be a maximal ideal on T containing $\mathscr{I}_Z$ and $\{T - Z\}$. We claim that

$$f[\mathscr{M}] \neq \mathscr{M} \neq f^2[\mathscr{M}].$$

Indeed, if $f[\mathscr{M}] = \mathscr{M}$, then, by Theorem 9.2 (a) of [2], we have $\{x \in T \colon f(x) \neq x\} \in \mathscr{M}$. But this is impossible, because

$$Z \notin \mathscr{M} \quad \text{and} \quad Z \subseteq \{x \in T \colon f(x) \neq x\}.$$

In the same way, if $f^2[\mathscr{M}] = \mathscr{M}$, then

$$\{x \in T \colon f^2(x) \neq x\} \in \mathscr{M}.$$

But then again $Z \subseteq \{x \in T \colon f^2(x) \neq x\}$, which contradicts $Z \notin \mathscr{M}$.

Choose now $X \subseteq Z$ such that $X \notin \mathscr{M}$ and $X \in f[\mathscr{M}]$ and $X \in f^2[\mathscr{M}]$. Let

$$Y = X - (f^{-1}[X] \cup f^{-2}[X]).$$

Then again $Y \notin \mathscr{M}$. But now $Y \cap f[Y] = Y \cap f^2[Y] = 0$. Moreover, $Y \notin \mathscr{I}_Z$; thus, by Proposition 3 (iii), $Y \notin \mathscr{F}$.

Consider a partition $\mathscr{V} = \{\{x, f(x)\} \colon x \in Y\}$ of $Y \cup f[Y]$. By Proposition 1, (a) $\Leftrightarrow$ (c), there is a selector S of $\mathscr{V}$ in $\mathscr{F}$. Since $S \cup f[S] \supseteq f[Y]$, we see that $S \notin \mathscr{I}_{\mathscr{F}}$. Thus we can choose $V \subseteq S$ such that $V \notin \mathscr{F}$. But then $f[V] \subseteq f[S]$ and $S \cap f[S] = 0$. Consequently, V and $f[V]$ are $\mathscr{F}$-separable sets. By Lemma 1, $V \in \mathscr{F}$, which yields a contradiction.

Proof of the Theorem. Suppose f is a non-trivial automorphism of $\mathscr{F}$. Let $T = A \cup B \cup C$, where

$$A = \{x \in T\colon f(x) = x\},$$
$$B = \{x \in T\colon f(x) \neq x \text{ and } f^2(x) = x\},$$
$$C = \{x \in T\colon f(x) \neq x \text{ and } f^2(x) \neq x\}.$$

To get a contradiction, suppose $\mathscr{F} \in S(2)$. Since f is non-trivial, we have $B \cup C \notin \mathscr{I}_{\mathscr{F}}$, and, by Lemma 2, $C \in \mathscr{I}_{\mathscr{F}}$. Thus $B \notin \mathscr{I}_{\mathscr{F}}$. We can decompose B into two disjoint sets D and E such that $f[D] = E$. Of course, $D \notin \mathscr{I}_{\mathscr{F}}$. Otherwise, we would have $E = f[D] \in f[\mathscr{I}_{\mathscr{F}}] = \mathscr{I}_{\mathscr{F}}$, and thus $B = D \cup E \in \mathscr{I}_{\mathscr{F}}$, which is impossible. Consider a partition

$$\mathscr{U} = \{\{x, f(x)\}\colon x \in D\}$$

of B. By Proposition 1, (a) $\Leftrightarrow$ (c), we have a selector S of $\mathscr{U}$ in $\mathscr{F}$. Notice that $S \notin \mathscr{I}_{\mathscr{F}}$. Otherwise, we would have $S \cup f[S] = B \in \mathscr{I}_{\mathscr{F}}$, which is impossible. Thus choose $X \subseteq S$ such that $X \notin \mathscr{F}$. But then X and $f[X]$ are $\mathscr{F}$-separable and, by Lemma 1, we have $X \in \mathscr{F}$, a contradiction. This completes the proof of the Theorem.

REFERENCES

[1] J. Cichoń and M. Porada, *Automorphisms of fields of sets*, Bull. Acad. Polon. Sci., Sér. Sci. Math., 29 (1981), pp. 435–437.

[2] W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, Berlin–Heidelberg–New York 1974.

[3] B. Węglorz, *Large invariant ideals on algebras*, Algebra Universalis (in print).

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