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ON FIXED POINT THEOREMS FOR MULTIFUNCTIONS IN DENDROIDS

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1. Introduction. The space X considered in this paper is always assumed to be an arbitrary *dendroid*, i.e., a hereditarily unicoherent and arcwise connected metric continuum. The *hereditary unicoherence* of X means that for every two subcontinua K and L of X the common part $K \cap L$ is a continuum. The *arcwise connectedness* of X states that for any two points a and b of X there exists in X an arc joining these points a and b. By the hereditary unicoherence of X, this is the unique arc between a and b in X, and so it can be denoted by ab.

The *multifunctions* F of X into itself will be considered which assign to each point $x \in X$ a non-empty closed subset $F(x)$ of X. Such a multifunction F is called *upper semi-continuous* if for every closed subset $A \subset X$ the upper counter-image $F^{-1}(A) = \{x \in X\colon F(x) \cap A \neq \emptyset\}$ is closed. F is called *lower semi-continuous* provided that the counter-image $F^{-1}(A)$ is open whenever A is open. For the metric compact space X the upper semi-continuity of F means that for every convergent sequence of points $x_n \in X$ $(n = 1, 2, \ldots)$ the inclusion

$$\operatorname*{Ls}_n F(x_n) \subset F(\lim_n x_n)$$

holds (see, e.g., [2], p. 61) and the lower semi-continuity of F means that

$$F(\lim_n x_n) \subset \operatorname*{Li}_n F(x_n)$$

(see, e.g., [2], p. 62).

A point $x \in X$ is called a *fixed point* of a multifunction F of X into itself if $x \in F(x)$. Two classes of multifunctions F of X into itself are the most important for the theorem stating that there exists a fixed point of F. First, upper semi-continuous multifunctions F, called *c-functions*, which assign to each $x \in X$ a subcontinuum $F(x)$ of the continuum X. Second, multifunctions F of X into itself which are *continuous*, i.e., both upper and lower semi-continuous. In these two cases the fixed point theorem for a dendroid X was proved by Ward, Jr. (see [4] and [5]), who used an order theoretical method by a characterization of an arbitrary dendroid X in terms of ordered sets.

The aim of the present paper is to give other proofs of those two theorems which both will be a modification of the method which I have developed in my fixed point theorem [3]. Such a modification will give, may be, a direction for a generalization of the theorem of Ward, Jr. [4] (for continuous multifunctions) from dendroids to λ-dendroids, i.e., hereditarily unicoherent and λ-connected continua (see [1]).

2. Preliminaries. For any two arcs ab, $ac \subset X$ with the same initial point a, we define the association $ab \prec ac$ (see [3], p. 106) if $ab \cap ac \neq (a)$, i.e., if $ab \cap ac$ is an arc non-degenerate to one point a. The association $\prec$ is an equivalence relation in the family of all arcs with the same initial point (see [3], p. 108, Proposition 2) and, of course, $ab \subset ac$ implies $ab \prec ac$ for a non-degenerate arc ab.

Since, by the hereditary unicoherence of X, for every continuum $K \subset X$

(2.1) $b, c \in K$ implies $bc \subset K$,

we have

(2.2) $a \notin K$ and $b, c \in K$ imply $ab \prec ac$

and, moreover (see [3], p. 108, Proposition 6),

(2.3) $ab \subset ad$ and $bc \prec bd$ imply $ab \subset ac$.

It is worth noting that, by (2.1), $b \in ac$ means that $ab \subset ac$, which is equivalent to the equality $ab \cup bc = ac$.

If F is a c-function mapping X into itself, then the image $F(K)$, defined by the formula

$$F(K) = \bigcup_{x \in K} F(x) \quad \text{for any } K \subset X,$$

satisfies the following condition:

(2.4) $F(K)$ is a continuum whenever K is a continuum

(see, e.g., [5], p. 161).

If F is a continuous multifunction in X, then for every component $\mathscr{C}F(K)$ of the image $F(K)$ of an arbitrary continuum $K \subset X$ and for every $x \in K$ the inequality $F(x) \cap \mathscr{C}F(K) \neq \emptyset$ holds (see [4], p. 924). It follows that for any two continua K, $L \subset X$ and for every component $\mathscr{C}F(K)$

(2.5) $K \cap L \neq \emptyset$ implies that there exists a component $\mathscr{C}F(L)$ such that $\mathscr{C}F(K) \cap \mathscr{C}F(L) \neq \emptyset$.

3. The fixed point theorem for c-functions. The following theorem, due to Ward, Jr. [5], will now be proved:

THEOREM 1. *If F is a c-function mapping a dendroid X into itself, then there exists a fixed point of F.*

Proof. For the fixed point, consider the family $\mathscr{P}_a$ of all arcs $ab \subset X$ defined as follows: for every $p \in ab-(b)$ and for every $q \in F(p)$ the association $pq \prec pb$ holds.

The following four properties of families $\mathscr{P}_a$ will yield the proof of the theorem.

(3.1) *If $a \notin F(a)$, then for every $d \in F(a)$ there exists $ab \in \mathscr{P}_a$ such that $ab \subset ad$.*

For the proof of (3.1), assume that $d \in F(a)$ and that an arc $ab \subset ad$ satisfies, by the upper semi-continuity of F, the equality $ab \cap F(ab) = \emptyset$. Then for every $p \in ab-(b)$ and every $q \in F(p)$ we have $p \notin F(ab)$ and $q, d \in F(ab)$. Since $F(ab)$ is a continuum by (2.4), we have $pq \prec pd$ by (2.2). But $pd \prec pb$ since $ab \subset ad$, and therefore $pq \prec pb$ by the transitivity of association.

(3.2) *If $ab \in \mathscr{P}_a$, $b \notin F(b)$, and $d \in F(b)$, then $ab \subset ad$.*

Since $b \notin F(b)$ by assumption, the upper semi-continuity of F implies that

$$p'b \cap F(p'b) = \emptyset \qquad \text{for some } p' \in ab-(b).$$

For every $p \in p'b-(b)$ we have $p \in ab-(b)$, whence $ap \subset ab$ and $pb \prec pq$ for every $q \in F(p)$ by the definition of $\mathscr{P}_a$. From (2.3) it follows that $ap \subset aq$. But $p \notin F(p'b)$ and $q, d \in F(p'b)$, whence $pq \prec pd$ by (2.2) and (2.4). Therefore, by (2.3), $ap \subset ad$ for every $p \in p'b-(b)$. Since the union of all such arcs ap is $ab-(b)$, we have $ab-(b) \subset ad$, and hence $ab \subset ad$.

(3.3) *If $ab \cup bc = ac$, $ab \in \mathscr{P}_a$ and $bc \in \mathscr{P}_b$, then $ac \in \mathscr{P}_a$.*

Indeed, if $p \in ab-(b)$, then $pc \prec pb$, since $ab \subset ac$ by assumption. Moreover, $pb \prec pq$ for every $q \in F(p)$ because $ab \in \mathscr{P}_a$. Hence $pq \prec pc$ by the transitivity of association. If $p \in bc-(c)$, then $pc \prec pq$ by the assumption that $bc \in \mathscr{P}_b$.

(3.4) *If $ab = \overline{\bigcup_{n=1}^{\infty} ab_n}$ and $ab_n \in \mathscr{P}_a$ $(n = 1, 2, \ldots)$, then $ab \in \mathscr{P}_a$.*

Let $p \in ab-(b)$. Then there exists an n such that $p \in ab_n-(b_n)$. Therefore, $pb_n \prec pb$ since $ab_n \subset ab$, and $pq \prec pb_n$ for every $q \in F(p)$ by the assumption that $ab_n \in \mathscr{P}_a$. From the reflexivity of association it follows that $pq \prec pb$.

To complete the proof of Theorem 1 observe that in the dendroid X for every increasing sequence of arcs $ab_1 \subset ab_2 \subset \ldots$ the closure $\overline{\bigcup_{n=1}^{\infty} ab_n}$ is an arc ab (a short proof of this statement is given in [3], p. 109, Remark 2). Thus, by (3.1) and (3.4), and by a version of the Brouwer reduction theorem (see [3], p. 115), for $a \notin F(a)$ there exists an arc ab maximal in $\mathscr{P}_a$. Then from (3.1)–(3.3) it follows that $b \in F(b)$.

Indeed, by (3.2), if $b \notin F(b)$, then $ab \subset ad$ for every $d \in F(b)$, and, by (3.1), there exists $bc \subset bd$ such that $bc \in \mathscr{P}_b$. Thus $ab \cup bc = ac \in \mathscr{P}_a$ by (3.3), which contradicts the maximality of ab in $\mathscr{P}_a$.

4. The fixed point theorem for continuous multifunctions. Let F be an arbitrary continuous multifunction of the dendroid X into itself. For an arbitrary arc $ab \subset X$ a sequence of components $\mathscr{C}F(b_{i-1} b_i)$ of $F(b_{i-1} b_i)$ for $i = 1, 2, \ldots$ $(b_0 = a)$ will be denoted by $[\mathscr{C}F(a), \mathscr{C}F(b)]$ provided that the arcs ab_i constitute a countable partition of ab, i.e.,

$$ab_{i-1} \subsetneqq ab_i \text{ for every } i \quad \text{and} \quad \bigcup_{i=1}^{\infty} ab_i = ab-(b),$$

and that the following chain conditions hold:

$$\mathscr{C}F(a) \subset \mathscr{C}F(ab_1), \qquad \mathscr{C}F(b_{i-1} b_i) \cap \mathscr{C}F(b_i b_{i+1}) \neq \emptyset,$$

$$\operatorname*{Ls}_i \mathscr{C}F(b_{i-1} b_i) \cap \mathscr{C}F(b) \neq \emptyset.$$

Then we say that $ab \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$ if there exists a chain $[\mathscr{C}F(a), \mathscr{C}F(b)]$ such that for every $i = 1, 2, \ldots$ and every $p \in b_{i-1} b_i - (b_i)$ there exists $q \in \mathscr{C}F(b_{i-1} b_i)$ with $pq \prec pb$.

LEMMA 1. *If $a \notin F(a)$, then for every component $\mathscr{C}F(a)$ and every $d \in \mathscr{C}F(a)$ there exist an arc $ab \subset ad$ and a component $\mathscr{C}F(b)$ such that $ab \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$.*

Proof. Let

$$d \in \mathscr{C}F(a) \tag{4.1}$$

and let

$$ab \subset ad \tag{4.2}$$

be an arc such that $ab \cap F(ab) = \emptyset$ by the upper semi-continuity of F. Then taking the component $\mathscr{C}F(ab)$ of $F(ab)$ which contains $\mathscr{C}F(a)$ we have

$$ab \cap \mathscr{C}F(ab) = \emptyset, \tag{4.3}$$

$$\mathscr{C}F(a) \subset \mathscr{C}F(ab). \tag{4.4}$$

Take an arbitrary countable partition $\overline{\bigcup_{i=1}^{\infty} ab_i}$ of ab and define a chain $[\mathscr{C}F(a), \mathscr{C}F(b)]$ for some $\mathscr{C}F(b)$ as follows:

Let $\mathscr{C}F(ab_1)$ denote the component of $F(ab_1)$ which contains $\mathscr{C}F(a)$ Take, by (2.5), a component $\mathscr{C}F(b_1 b_2)$ of $F(b_1 b_2)$ such that

$$\mathscr{C}F(b_1 b_2) \cap \mathscr{C}F(ab_1) \neq \emptyset$$

and, by induction, let $\mathscr{C}F(b_i b_{i+1}) \cap \mathscr{C}F(b_{i-1} b_i) \neq \emptyset$. Since $b_{i-1} b_i \to (b)$, by the upper semi-continuity we obtain

$$\operatorname*{Ls}_i \mathscr{C}F(b_{i-1} b_i) \subset F(b).$$

Thus for $\mathscr{C}F(b)$ we take in $F(b)$ the component of an arbitrary point of $\operatorname*{Ls}_i \mathscr{C}F(b_{i-1} b_i)$.

Since the union $\bigcup_{i=1}^{\infty} \mathscr{C}F(b_{i-1} b_i)$ is connected and contains $\mathscr{C}F(a)$, by (4.4) we get

$$\bigcup_{i=1}^{\infty} \mathscr{C}F(b_{i-1} b_i) \subset \mathscr{C}F(ab).$$

Now, let $p \in b_{i-1} b_i - (b_i)$. Then for an arbitrary $q \in \mathscr{C}F(b_{i-1} b_i)$ we have $q \in \mathscr{C}F(ab)$. Moreover, $d \in \mathscr{C}F(ab)$ by (4.1) and (4.4), and $p \notin \mathscr{C}F(ab)$ by (4.3). It follows from (2.2) that $pq \prec pd$. But $pd \prec pb$ in view of (4.2), and therefore $pq \prec pb$ by the transitivity of association.

LEMMA 2. *If* $ab \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$, $d \in \mathscr{C}F(b)$ *and* $b \notin F(b)$, *then* $ab \cup bd = ad$.

Proof. Let $p'b \cap F(p'b) = \emptyset$ for some $p' \in ab-(b)$ by the assumption that $b \notin F(b)$ and the upper semi-continuity of F. Then taking the component $\mathscr{C}F(p'b)$ which contains the union $\bigcup_{i=i_0}^{\infty} \mathscr{C}F(b_{i-1} b_i)$, where i_0 is the first integer such that $b_{i_0-1} \in p'b$, we have

$$p'b \cap \mathscr{C}F(p'b) = \emptyset, \tag{4.5}$$

$$\bigcup_{i=i_0}^{\infty} \mathscr{C}F(b_{i-1} b_i) \subset \mathscr{C}F(p'b). \tag{4.6}$$

Moreover, we have

$$\operatorname*{Ls}_i \mathscr{C}F(b_{i-1} b_i) \cap \mathscr{C}F(b) \neq \emptyset \quad \text{and} \quad \operatorname*{Ls}_i \mathscr{C}F(b_{i-1} b_i) \subset \mathscr{C}F(p'b),$$

and therefore by (4.6) we obtain

$$\mathscr{C}F(b) \subset \mathscr{C}F(p'b). \tag{4.7}$$

Let $p \in p'b-(b) \subset ab-(b)$. Then $p \in b_{i-1} b_i - (b_i)$ for some $i \geqslant i_0$, and for some $q \in \mathscr{C}F(b_{i-1} b_i)$ we have $pq \prec pb$ by the definition of $\mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$. But $p \in ab$ implies $ap \subset ab$, and hence $ap \subset aq$ by (2.3).

Simultaneously, $q \in \mathscr{C}F(p'b)$ by (4.6), $d \in \mathscr{C}F(p'b)$ by (4.7), and $p \notin \mathscr{C}F(p'b)$ by (4.5), whence $pq \prec pd$ by (2.2). It follows from (2.3) that $ap \subset ad$ for every $p \in p'b-(b)$. Since the union of all such arcs ap is $ab-(b)$, we get $ab-(b) \subset ad$. Therefore $ab \subset ad$, i.e., $ab \cup bd = ad$.

LEMMA 3. *Let* $ab \cup bc = ac$. *If*

$$ab \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)] \quad \text{and} \quad bc \in \mathscr{P}_b[\mathscr{C}F(b), \mathscr{C}F(c)],$$

then $ac \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(c)]$ *with a chain* $[\mathscr{C}F(a), \mathscr{C}F(c)]$ *joining* $\mathscr{C}F(b)$ *and* $\mathscr{C}F(c)$ *(which means that* $\mathscr{C}F(b) \subset \mathscr{C}F(c_{i-1}c_i)$ *for some* i*).*

Proof. We will construct the desired chain $[\mathscr{C}F(a), \mathscr{C}F(c)]$ replacing all components $\mathscr{C}F(b_{i-1}b_i)$ of the chain $[\mathscr{C}F(a), \mathscr{C}F(b)]$ by the component $\mathscr{C}F(ab)$ such that

$$\bigcup_{i=1}^{\infty} \mathscr{C}F(b_{i-1}b_i) \subset \mathscr{C}F(ab). \tag{4.8}$$

Since

$$\operatorname*{Ls}_i \mathscr{C}F(b_{i-1}b_i) \cap \mathscr{C}F(b) \neq \emptyset \quad \text{and} \quad \mathscr{C}F(b) \subset \mathscr{C}F(bc_1),$$

we have $\mathscr{C}F(ab) \cap \mathscr{C}F(bc_1) \neq \emptyset$. Therefore, $\mathscr{C}F(ab)$, $\mathscr{C}F(bc_1)$, $\mathscr{C}F(c_1c_2), \ldots$ constitute the chain $[\mathscr{C}F(a), \mathscr{C}F(c)]$ joining $\mathscr{C}F(b)$ and $\mathscr{C}F(c)$.

Now, to prove that $ac \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$, let $p \in ab-(b)$. Then there exists an i such that $p \in b_{i-1}b_i-(b_i)$, and hence $pb \prec pq$ for some $q \in \mathscr{C}F(b_{i-1}b_i)$. Simultaneously, $pc \prec pb$, since $ab \subset ac$ by assumption, and therefore $pq \prec pc$ by the transitivity of association $\prec$. Moreover, $q \in \mathscr{C}F(ab)$ by (4.8).

LEMMA 4. *If*

$$ab = \overline{\bigcup_{n=1}^{\infty} ab_n},$$

where $ab_n \subsetneqq ab_{n+1}$ *and* $ab_n \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b_n)]$ *for every* $n = 1, 2, \ldots$ *and if the chain* $[\mathscr{C}F(a), \mathscr{C}F(b_{n+1})]$ *joins* $\mathscr{C}F(b_n)$ *and* $\mathscr{C}F(b_{n+1})$ *for every* $n = 1, 2, \ldots$, *then* $ab \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$ *for some component* $\mathscr{C}F(b)$.

Proof. For the given partition $ab-(b) = \bigcup_{n=1}^{\infty} ab_n$ we will define the desired chain $[\mathscr{C}F(a), \mathscr{C}F(b)]$ of components $\mathscr{C}F(b_nb_{n+1})$ for $n = 0, 1, 2, \ldots$ $(b_0 = a)$.

Take, for $\mathscr{C}F(ab_1)$, the component of $F(ab_1)$ which contains the union $\bigcup_{i=1}^{\infty} \mathscr{C}F(b_{i-1}b_i)$ of the components of the chain $[\mathscr{C}F(a), \mathscr{C}F(b_1)]$. Then, obviously, $\mathscr{C}F(a) \subset \mathscr{C}F(ab_1)$. Now assume, by induction, that a sequence of components $\mathscr{C}F(ab_1)$, $\mathscr{C}F(b_1b_2), \ldots, \mathscr{C}F(b_{n-1}b_n)$ is defined so that

$$\mathscr{C}F(b_{i-1}b_i) \cap \mathscr{C}F(b_ib_{i+1}) \neq \emptyset \quad \text{and} \quad \mathscr{C}F(b_ib_{i+1}) \cap \mathscr{C}F(b_{i+1}) \neq \emptyset$$

for $i = 1, 2, \ldots, n-1$. Let $\mathscr{C}F(b_nb_{n+1})$ denote the component of $F(b_nb_{n+1})$

which contains the union $\bigcup_{i=i_n}^{\infty} \mathscr{C}F(b_{n+1,i-1}\, b_{n+1,i})$ of some components of the chain $[\mathscr{C}F(a), \mathscr{C}F(b_{n+1})]$, where i_n is such that

$$\mathscr{C}F(b_n) \subset \mathscr{C}F(b_{n+1,i_n-1}\, b_{n+1,i_n}).$$

Then

$$\mathscr{C}F(b_{n-1}\, b_n) \cap \mathscr{C}F(b_n b_{n+1}) \neq \emptyset, \qquad \mathscr{C}F(b_{n+1}) \cap \mathscr{C}F(b_n b_{n+1}) \neq \emptyset$$

and

$$\bigcup_{i=i_n}^{\infty} \mathscr{C}F(b_{n+1,i-1}\, b_{n+1,i}) \subset \mathscr{C}F(b_n b_{n+1}). \tag{4.9}$$

The sequence of components $\mathscr{C}F(b_n b_{n+1})$ so defined has the property that

$$\operatorname*{Ls}_n \mathscr{C}F(b_n b_{n+1}) \cap \mathscr{C}F(b) \neq \emptyset$$

by the upper semi-continuity of F. Thus this sequence forms a chain $[\mathscr{C}F(a), \mathscr{C}F(b)]$.

Now, let $p \in b_n b_{n+1} - (b_{n+1})$. Then $p \in b_{n+1,i-1}\, b_{n+1,i} - (b_{n+1,i})$ for some $i \geqslant i_n$, and therefore there exists $q \in \mathscr{C}F(b_{n+1,i-1}\, b_{n+1,i})$ such that $pq \prec pb_{n+1}$. Since $pb_{n+1} \prec pb$, we have $pq \prec pb$ by the transitivity of association $\prec$. By (4.9) we obtain $q \in \mathscr{C}F(b_n b_{n+1})$.

The theorem of Ward, Jr. [4], can now be proved.

THEOREM 2. *If F is a continuous multifunction mapping a dendroid X into itself, then there exists a fixed point of F.*

Proof. Since for every increasing sequence of arcs $ab_1 \subsetneqq ab_2 \subsetneqq \ldots$ the closure $\overline{\bigcup_{n=1}^{\infty} ab_n}$ is an arc ab in the dendroid X, by Lemmas 1 and 4 and by a version of the Brouwer reduction theorem for $a \notin F(a)$ there exists an arc ab which is maximal in $\mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(b)]$ in the sense that there exists no greater arc $ac \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(c)]$ with the chain $[\mathscr{C}F(a), \mathscr{C}F(c)]$ joining $\mathscr{C}F(b)$ and $\mathscr{C}F(c)$. Then from Lemmas 1–3 it follows that $b \in F(b)$ for the maximal arc ab.

Indeed, if $b \notin F(b)$, then $ab \subset ad$ for every $d \in \mathscr{C}F(b)$ by Lemma 2, and by Lemma 1 there exist an arc $bc \subset bd$ and a component $\mathscr{C}F(c)$ such that $bc \in \mathscr{P}_b[\mathscr{C}F(b), \mathscr{C}F(c)]$ for some chain $[\mathscr{C}F(b), \mathscr{C}F(c)]$. Thus, by Lemma 3, $ac \in \mathscr{P}_a[\mathscr{C}F(a), \mathscr{C}F(c)]$ with the chain $[\mathscr{C}F(a), \mathscr{C}F(c)]$ joining $\mathscr{C}F(b)$ and $\mathscr{C}F(c)$, which contradicts the maximality of ab.

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