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MIXTURES OF NON-ATOMIC MEASURES. III

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1. Introduction. In part I of the series by the first-named author and M. Bhaskara Rao [1] a partial solution of the problem, what transition functions have every mixture to be non-atomic, is presented. In part II [3] we have given a complete solution of the problem, what spaces have every mixture of non-atomic measures to be non-atomic. We now reconsider the problem of [1] and give a neat and complete solution. Results of [1] will follow from the present ones.

2. Notation. $(X, \boldsymbol{A})$ and $(Y, \boldsymbol{B})$ are two fixed Borel spaces, and Q is a transition function on $X \times \boldsymbol{B}$. We consider only probability measures. Q is *non-atomic* if, for all $x \in X$, $Q(x, \cdot)$ is non-atomic on $\boldsymbol{B}$. The function Q is *uniformly non-atomic* if there exists a countably generated $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that, for all $x \in X$, $Q(x, \cdot)$ is non-atomic on $\boldsymbol{B}_0$. The word "uniformly" is used since, for a single measure, non-atomicity means the same as uniform non-atomicity as noted in [2]. Let λ be a measure on $\boldsymbol{A}$. The function Q is *λ-uniformly non-atomic* if there is an $A \in \boldsymbol{A}$ with $\lambda(A) = 1$ such that Q restricted to $A \times \boldsymbol{B}$ is uniformly non-atomic, that is, there is a countably generated $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that, for all $x \in A$, $Q(x, \cdot)$ is non-atomic on $\boldsymbol{B}_0$. The function Q is *λ-functionally non-atomic* if the vector measure

$$Q^*\colon \boldsymbol{B} \to L_1(X, \boldsymbol{A}, \lambda),$$

given by $Q^*(B) = Q(\cdot, B)$, is non-atomic, that is, $Q^*(B) > 0$ implies that there is a $C \in \boldsymbol{B}$, $C \subset B$, with $0 < Q^*(C) < Q^*(B)$. Here, as usual, $f \leqslant g$ means that $\lambda(x\colon f(x) \leqslant g(x)) = 1$ and $f < g$ means that $f \leqslant g$ but $f \neq g$. The function Q is *dominated* if there is a measure ν on $\boldsymbol{B}$ such that, for every x, $Q(x, \cdot) \ll \nu$. The function Q is *continuous* if X is a topological space and, for every $B \in \boldsymbol{B}$, $Q(\cdot, B)$ is a continuous function on X. The *mixture* of Q with respect to λ is the measure μ on $\boldsymbol{B}$ defined by

$$\mu(B) = \int Q(x, B)\, d\lambda(x).$$

3. Main results. Consider the following statements:

(I) *Q is λ-uniformly non-atomic.*

(II) *Q is λ-functionally non-atomic.*

(III) *The mixture of Q with respect to λ is non-atomic.*

THEOREM. (a) (I) $\Rightarrow$ (II) *and* (II) $\Leftrightarrow$ (III).

(b) (III) *does not imply* (I) *even if Q is non-atomic.*

Proof of (a). (I) $\Rightarrow$ (II). Let $\boldsymbol{B}_0 \subset \boldsymbol{B}$ and $A \in \boldsymbol{A}$ witness that Q is λ-uniformly non-atomic. Let $\{B_i, i \geqslant 1\}$ be a countable field generating $\boldsymbol{B}_0$. To see that Q is λ-functionally non-atomic, assume that $B \in \boldsymbol{B}$ is given with $Q^*(B) > 0$. If, for every i,

$$Q^*(B \cap B_i) = 0 \quad \text{or} \quad Q^*(B \cap B_i) = Q^*(B),$$

we obtain a contradiction. Indeed, let

$$C_i = \begin{cases} B_i & \text{if } Q^*(B \cap B_i) = Q^*(B), \\ B_i^c & \text{if } Q^*(B \cap B_i) = 0. \end{cases}$$

Then

$$C = \bigcap_{i \geqslant 1} C_i$$

is an atom of $\boldsymbol{B}_0$ and

$$Q^*(C) \geqslant Q^*(B \cap C) = Q^*(B) > 0.$$

Thus, for at least one $x \in A$, we have $Q(x, C) > 0$ contradicting the fact that, for all $x \in A$, $Q(x, \cdot)$ is non-atomic on $\boldsymbol{B}_0$.

(II) $\Leftrightarrow$ (III). The mixture μ of Q with respect to λ is given by

$$\mu(B) = \int Q^*(B)(x)\, d\lambda(x) \qquad \text{for } B \in \boldsymbol{B}.$$

Thus $\mu(B) > 0$ iff $Q^*(B) > 0$. Moreover, for $C \subset B$, $C \in \boldsymbol{B}$, $\mu(C) < \mu(B)$ iff $Q^*(C) < Q^*(B)$. Thus the proof of (a) is completed.

Proof of (b). We give an example where (III) holds but (I) does not hold. Let $\aleph$ be an uncountable cardinal and let $Y = \{0, 1\}^{\aleph}$ with $\boldsymbol{B}$ being the product σ-algebra. Let X be the set of all those points of $\{0, 1, 2\}^{\aleph}$ which have infinitely many 2's and let $\boldsymbol{A}$ be the restriction of the product σ-algebra to X, each factor space having a discrete σ-algebra. Let λ be the product measure on $\{0, 1, 2\}^{\aleph}$, where each coordinate space has measure $\frac{1}{2}$ at 0 and $\frac{1}{2}$ at 1. The outer measure of X with respect to λ being 1, we can restrict λ to X and we denote it still by λ. For each $x \in X$, let $Q(x, \cdot)$ be the product measure on Y, where the a-th coordinate has mass concentrated at x_a if $x_a = 0$ or 1 and is $\frac{1}{2}, \frac{1}{2}$ measure if $x_a = 2$. Let μ be the measure on Y given by the product of $\frac{1}{2}, \frac{1}{2}$ measures on factor spaces. It can be verified that Q is a non-atomic transition function on $X \times \boldsymbol{B}$ and the mixture of Q with respect to λ is μ, and hence is non-atomic. We now show that Q is not λ-uniformly non-atomic. In fact, a stronger statement is valid:

Let $\boldsymbol{B}_0 \subset \boldsymbol{B}$ be any countably generated sub-σ-algebra of $\boldsymbol{B}$ and let

$$A = \{x\colon Q(x, \cdot) \text{ is non-atomic on } \boldsymbol{B}_0\}.$$

Then $\lambda(A) = 0$.

Indeed, if $\boldsymbol{B}_0$ depends only on the first a coordinates of Y, then

$$A \subset \{x\colon x_i = 2 \text{ for infinitely many } i \leqslant a\}$$

which has λ-measure zero.

COROLLARY 1 (Theorem 1 of [1]). *If Q is non-atomic and $\boldsymbol{B}$ is countably generated, then any mixture of Q is non-atomic.*

Indeed, then Q is uniformly non-atomic, and hence is λ-uniformly non-atomic for any λ, and so the Theorem applies.

COROLLARY 2 (Theorem 2 of [1]). *If Q is non-atomic and dominated, then any mixture of Q is non-atomic.*

Indeed, the dominating measure ν can be taken to be non-atomic, and so there is a countably generated $\boldsymbol{B}_0 \subset \boldsymbol{B}$ on which ν is non-atomic [2]. Consequently, Q is λ-uniformly non-atomic for any λ, and so the Theorem applies.

COROLLARY 3 (Theorem 3 of [1]). *If Q is non-atomic and continuous and X has a countable dense set, then any mixture of Q is non-atomic.*

Indeed, if $x_1, x_2, \ldots$ is a dense set in X, then

$$\nu(B) = \sum \frac{Q(x_i, B)}{2^i}$$

dominates Q, and so Corollary 2 can be applied.

Similarly we obtain Theorem 4 of [1].

COROLLARY 4 (Theorem 2.2.7 of [4]). *Let X be a metric space with a dense set whose cardinal is less than the first measurable cardinal and let $\boldsymbol{A}$ be the set of Borel subsets of X. If Q is continuous and non-atomic, then any mixture of Q is non-atomic.*

Indeed, any λ on such a space is concentrated on a separable subset, and so Corollary 3 applies to prove that Q is λ-uniformly non-atomic.

Remark. It is not true that if λ and Q are non-atomic, then the mixture is non-atomic. We give one such example following the ideas of [5]. Consider $X, \boldsymbol{A}$ and λ to be the unit interval, the Borel field and the Lebesgue measure, respectively. Let $Y = X \times X$ and let $D \subset [0, 1] \times [0, \frac{1}{2}]$ be a set of outer planar Lebesgue measure $\frac{1}{2}$ which contains at most one point from each vertical line. Let $\boldsymbol{B}$ be all planar Borel subsets of Y which either contain D or are disjoint from D. For each $x \in X$, let $Q(x, \cdot)$ be the linear Lebesgue measure on the x-section of D^c. Clearly, Q and λ are non-atomic but the mixture μ is not, since, for any $B \in \boldsymbol{B}$, if $B \supset D$, then $\mu(B) = \frac{1}{2}$.

Of course, if λ and Q are non-atomic, then the mixture cannot be 0-1 valued in a sense made precise in Remark 3 of [3].

REFERENCES

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Mixtures of nonatomic measures*, Proceedings of the American Mathematical Society 33 (1972), p. 507-510.

[2] — *A remark on nonatomic measures*, The Annals of Mathematical Statistics 43 (1972), p. 369-370.

[3] K. P. S. Bhaskara Rao and B. V. Rao, *Mixtures of non-atomic measures. II*, Colloquium Mathematicum 33 (1975), p. 105-112.

[4] M. Bhaskara Rao, *Contributions to measure theory*, Thesis submitted to the Indian Statistical Institute, Calcutta 1971.

[5] D. Ramachandran, *Mixtures of perfect probability measures*, The Annals of Probability (to appear).

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