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RESTRICTED EULERIAN CIRCUITS IN DIRECTED GRAPHS

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1. Introduction. The purpose of this note is to characterize Eulerian circuits in directed graphs in which there is specified a preference as to which arcs leading from a vertex must be chosen for the circuit before other arcs may be chosen. Such circuits will be called restricted Eulerian circuits (for the definition see Section 2). The motivation for this definition is taken from the concept of a permutation of a multiset being a cycle, as defined in [2].

2. Definitions and notation. This section follows the terminology of [1].

A *directed graph* G consists of a set of vertices V and of a set of arcs A, where each arc e leads from an *initial vertex* $\mathrm{init}(e)$ to a *final vertex* $\mathrm{fin}(e)$. The case where $\mathrm{init}(e) = \mathrm{fin}(e)$ is not excluded, and several different arcs may have the same initial and final vertices.

If $v \in V$, then the *out-degree* of v is the number of arcs e such that $\mathrm{init}(e) = v$, and the *in-degree* of v is the number of arcs e such that $\mathrm{fin}(e) = v$. A graph G is *balanced* if for each $v \in V$ the in-degree of v equals the out-degree.

It will be assumed that G is *finite*, i.e., V and A are finite sets, and that G contains no *isolated vertices*, i.e., there is no vertex with in-degree and out-degree both equal to zero.

Let $f\colon A \to \{0, 1\}$ be defined as follows: for each $v \in V$ there is at least one $e \in A$ with $v = \mathrm{init}(e)$ such that $f(e) = 1$. The mapping f is called a *label* for G, and $f(e)$ is called the *f-label* of e. Thus each vertex of G has an arc leading from it f-labeled 1.

A *restricted Eulerian circuit* P of the graph G is a sequence of all the arcs $(e_1, e_2, \ldots, e_n)$ of A which satisfies

(i) $e_i \neq e_j$ for $i \neq j$;

(ii) $\mathrm{fin}(e_i) = \mathrm{init}(e_{i+1})$ for $1 \leqslant i \leqslant n-1$, and $\mathrm{fin}(e_n) = \mathrm{init}(e_1)$;

(iii) if $\mathrm{init}(e_i) = \mathrm{init}(e_j)$ and $f(e_i) < f(e_j)$, then $i < j$.

A restricted Eulerian circuit is a Eulerian circuit with the added condition (iii). If P is a restricted Eulerian circuit, it is said to *start* at $\mathrm{init}(e_1)$.

With the use of f, a relation $\precsim$ can be defined on V by $v_1 \precsim v_2$ if there exists a sequence of arcs $(e_1, e_2, \ldots, e_m)$ with $\mathrm{init}(e_1) = v_1$, $\mathrm{fin}(e_m) = v_2$, and $\mathrm{init}(e_i) = \mathrm{fin}(e_{i+1})$ for $1 \leqslant i \leqslant m-1$, where $f(e_i) = 1$ for $1 \leqslant i \leqslant m$. Note that there may exist $v \neq v'$ with $v \precsim v'$ and $v' \precsim v$. Let

$$V_M = \{v \in V\colon \text{ if } v \precsim v', \text{ then } v' \precsim v\}$$

be the set of ***maximal vertices*** of V. It is immediate that V_M is non-empty. If $v \precsim v'$ and $v' \precsim v$ for any distinct pair $v, v' \in V_M$, then V_M is said to be ***connected***.

It is obvious that $\precsim$ is transitive, and if $v \in V - V_M$, then there is a $v' \in V_M$ such that $v \precsim v'$. Thus there is no $w \in V_M$ such that $w \precsim v$.

3. A characterization of restricted Eulerian circuits. The main result of this note is

THEOREM 1. *Assume that G is a finite directed graph which is balanced and contains no isolated vertices, and f is a label for G. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) *there is a restricted Eulerian circuit for G.*

(2) *V_M is connected.*

Furthermore,

(3) *if V_M is connected, then any restricted Eulerian circuit must start at a vertex which is in V_M.*

Proof. To prove (3), suppose that $v \in V - V_M$ and that $P = (e_1, e_2, \ldots$ $\ldots, e_n)$ is a restricted Eulerian circuit which starts at v. Let i be the largest subscript such that $\mathrm{init}(e_i) \in V_M$ and $f(e_i) = 1$. Then $\mathrm{fin}(e_i) \in V_M$ and all the arcs leading from the vertices in V_M have subscripts less than or equal to i. Thus $i = n$ and $\mathrm{fin}(e_i) = v$, which implies $v \in V_M$, a contradiction.

To show that (1) implies (2), suppose that $v \in V_M$ and that the restricted Eulerian circuit $P = (e_1, e_2, \ldots, e_n)$ starts at v. Assume that V_M is not connected; then there is a $v' \in V_M$ such that $v \not\precsim v'$ and $v' \not\precsim v$. Let

$$W = \{w \in V_M\colon w \precsim v' \text{ and } v' \precsim w\}.$$

Let i be the largest subscript such that $\mathrm{init}(e_i) \in W$ and $f(e_i) = 1$. Since $\precsim$ is transitive, $\mathrm{fin}(e_i) \in W$. But all the arcs leading from $\mathrm{fin}(e_i)$ have subscripts less than or equal to i. Thus $i = n$ and $\mathrm{fin}(e_i) = v$, which implies $v \in W$, a contradiction.

To show that (2) implies (1), let $v \in V_M$. Since V_M is connected, for each $v' \in V$, $v' \neq v$, there is a sequence of arcs $(e_1, e_2, \ldots, e_m)$ with $f(e_i) = 1$ for $1 \leqslant i \leqslant m$, $\mathrm{init}(e_1) = v'$, $\mathrm{fin}(e_m) = v$, and $\mathrm{init}(e_{i+1}) = \mathrm{fin}(e_i)$ for $1 \leqslant i \leqslant m-1$. Let the order of v' be the length of the shortest sequence of such a kind.

For each vertex w of order 1, let e_w be an arc with $\text{init}(e_w) = w$, $\text{fin}(e_w) = v$, and $f(e_w) = 1$. Having chosen e_w for each vertex w of order less than k, let w be a vertex of order k. Now choose e_w so that $\text{init}(e_w) = w$, $\text{fin}(e_w)$ has the order $k-1$, and $f(e_w) = 1$. All of the arcs so chosen define an oriented tree in G (see [1]) with v as the root.

To define a restricted Eulerian circuit $P = (e_1, e_2, \ldots, e_n)$ starting at v, the arcs leading from each vertex w are partitioned into 3 classes: those f-labeled 0 comprise the class $(w, 1)$, those f-labeled 1 and not equal to e_w comprise the class $(w, 2)$, and e_w is the only member of the class $(w, 3)$.

Let e_1 be any member of $(v, 1)$ if $(v, 1)$ is not empty; otherwise, let e_1 be any member of $(v, 2)$ if $(v, 2)$ is not empty; otherwise, let $e_1 = e_v$. Suppose that the sequence $(e_1, e_2, \ldots, e_k)$ has been chosen and satisfies

(i) $\text{fin}(e_i) = \text{init}(e_{i+1})$ for $1 \leqslant i \leqslant k-1$;

(ii) for each i, $2 \leqslant i \leqslant k$,

$$e_i \in (\text{fin}(e_{i-1}), 1) - \{e_1, e_2, \ldots, e_{i-1}\}$$

if this set is not empty; otherwise,

$$e_i \in (\text{fin}(e_{i-1}), 2) - \{e_1, e_2, \ldots, e_{i-1}\}$$

if this set is not empty; otherwise,

$$e_i \in (\text{fin}(e_{i-1}), 3).$$

If $\text{fin}(e_k) = v$ and all the arcs leading out of v have not been chosen yet, or if $\text{fin}(e_k) \neq v$ and thus there exists at least one arc leading out of $\text{fin}(e_k)$ since G is balanced, then e_{k+1} is chosen so that

$$e_{k+1} \in (\text{fin}(e_k), 1) - \{e_1, e_2, \ldots, e_k\}$$

if this set is not empty; otherwise,

$$e_{k+1} \in (\text{fin}(e_k), 2) - \{e_1, e_2, \ldots, e_k\}$$

if this set is not empty; otherwise,

$$e_{k+1} \in (\text{fin}(e_k), 3).$$

If $\text{fin}(e_k) = v$ and all the arcs leading out of v have already been chosen, then all the arcs of A are in the sequence $P = \{e_1, e_2, \ldots, e_k\}$, and P is a restricted Eulerian circuit of G. This last follows since if all the arcs leading out of v have been chosen, then all those leading into v have been chosen. Thus all the arcs leading out of all the vertices of order 1 have been chosen. An induction argument shows that P contains all the arcs of A.

The construction above also gives

COROLLARY. *If V_M is connected, G is as in Theorem 1, and $v \in V_M$, then there is a restricted Eulerian circuit starting at v.*

4. Applications. If $g: A \to Z$, the integers, and if

$$n_v = \max\{g(e): \operatorname{init}(e) = v\},$$

then by letting

$$f(e) = \begin{cases} 0 & \text{if } g(e) < n_v \text{ for } v = \operatorname{init}(e), \\ 1 & \text{if } g(e) = n_v \text{ for } v = \operatorname{init}(e), \end{cases}$$

a similar theorem can be proved for directed graphs with more general labels such as g.

If for each vertex v the arcs leading out of v are labeled by mapping them one-to-one into consecutive positive integers starting with 1, then the resulting directed balanced graph and label are equivalent to a permutation of a multiset as described in [2]. The permutation is a cycle if the graph has a restricted Eulerian circuit. Thus Theorem 1 can be used to prove a generalization of Exercise 15 of Section 5.1.2 of [2].

If $f(e) = 1$ for each $e \in A$, then the existence of a restricted Eulerian circuit of G is equivalent to the existence of a Eulerian circuit on G, where G is considered simply as a directed graph. The well-known theorem of I. J. Good as stated in Section 2.3.4.2 of [1] is the following

THEOREM 2. *A finite directed graph with no isolated vertices has a Eulerian circuit if and only if it is connected and balanced.*

Thus, if G satisfies the conditions of Theorem 2 and $f(e) = 1$ for each arc e of G, then each vertex of G is a member of V_M. Hence Theorem 1 generalizes Theorem 2.

REFERENCES

[1] D. E. Knuth, *The art of computer programming*, Vol. I. *Fundamental algorithms*, 2nd edition, Reading, Mass., 1973.

[2] — *The art of computer programming*, Vol. III. *Sorting and searching*, Reading, Mass., 1973.

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P R O B L È M E S

P 21, R 2. La solution est positive ([1]).

I.2, p. 149, et XXXVII.1, p. 175.

([1]) E. Grzegorek, *Remarks on σ-fields without continuous measures*, this fascicle, p. 73-75.

ELŻBIETA POL (WARSZAWA)

P 1026. Formulé dans la communication *Strongly metrizable spaces of large dimension each separable subspace of which is zero-dimensional.*

Ce fascicule, p. 25.

J. MIODUSZEWSKI (KATOWICE)

P 1027 et **P 1028.** Formulés dans la communication *Compact Hausdorff spaces with two open sets.*

Ce fascicule, p. 35 et 39.

P 1027, R 1. As the author of the problem has informed us, E. van Douwen recently showed that the product of the Cantor set and of the double arrow line is a space with two open sets, thus answering the problem in the positive.

Letter of March 16, 1978.

MARCIN E. KUCZMA (WARSZAWA)

P 1029 et **P 1030.** Formulés dans la communication *Differentiation of implicit functions and Steinhaus' theorem in topological measure spaces.*

Ce fascicule, p. 106.

P. DIEROLF, S. DIEROLF (MUNICH) AND L. DREWNOWSKI (POZNAŃ)

P 1031 et **P 1032.** Formulés dans la communication *Remarks and examples concerning unordered Baire-like and ultrabarrelled spaces.*

Ce fascicule, p. 112 et 115.