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# LOCAL PROPERTIES OF COMPLETE BOOLEAN PRODUCTS

BY

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Products of complete Boolean algebras find application in the theory of models of the axiomatic set theory. The iteration of Boolean-valued models is closely connected with products of the underlying Boolean algebras. The usual algebraic definition of a complete product $\mathscr{B}$ of algebras $\mathscr{A}_0$, $\mathscr{A}_1$ requires the algebras $\mathscr{A}_0$, $\mathscr{A}_1$ to be independent in $\mathscr{B}$. However, as L. Bukovský pointed out in [2], from the set-theoretical point of view it seems more appropriate to consider the localized properties of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ in $\mathscr{B}$ than the total ones. The local independence (see the definition below) of factors $\mathscr{A}_0$, $\mathscr{A}_1$ in $\mathscr{B}$ is equivalent to the separation of the corresponding model classes. In the same way, the local disjointness of the factors is corelated with the disjointness of the model classes (Theorems 1, 2 in [2]).

In this paper we investigate the structure and the properties, especially the local ones, of complete Boolean products. As consequences we give answers to some questions concerning the $(\mathfrak{m}, 0)$-products in [7]. Other consequences are connected with models of the set theory. The paper is organized as follows.

In the introduction we bring the notion of complete Boolean product and its basic properties, especially in comparison with $(\mathfrak{m}, 0)$-products.

The first section is devoted to local properties. It is shown there that any complete product can be decomposed into its locally independent and locally nowhere independent part. The main result says that for independent products the local independence and the minimality are equivalent (Theorem 1.9). The final part of the section discusses the implications between the notions of independence, disjointness, local independence and local disjointness.

In the second section examples of complete Boolean products are given for two special algebras: Cantor algebra $\mathscr{C}$ and random algebra $\mathscr{R}$. Algebra $\mathscr{C}$ is formed from the field of all Borel subsets of the unit real interval factorized by the ideal $\mathscr{I}_{\mathscr{C}}$ of all meager sets. Analogous construction using the ideal $\mathscr{I}_{\mathscr{R}}$ of all sets of zero measure gives algebra $\mathscr{R}$. The main results of the section state that there is a product of algebras $\mathscr{R}$, $\mathscr{R}$, the so-called quadratic product, which is incomparable with the minimal product (Theorem 2.5). The quadratic product of $\mathscr{R}$, $\mathscr{R}$ is not locally independent, but it is locally disjoint.

The third section shows the connection of localized properties of complete products with properties of pairs of Cohen or random numbers.

The notions and denotations used in this paper follow, in general, the terminology used in [7]. However, there are some differences, e.g. we use $\wedge$, $\vee$, 0, 1 for Boolean operations and bound elements. By $h^{\mathscr{A}}$ we denote the projection of an algebra $\mathscr{B}$ onto its complete subalgebra $\mathscr{A}$, defined by $h^{\mathscr{A}}(X) = \bigwedge\{A \in \mathscr{A};\ A \geqslant X\}$, $X \in \mathscr{B}$. For set-theoretical notions see e.g. [6].

An ordinal we consider as the set of all lesser ordinals, a cardinal is an initial ordinal. The set of all functions from $A$ to $B$ is denoted by ${}^{A}B$. Thus, ${}^{\omega}2$ is the set of all infinite sequences of 0, 1. The set of all finite sequences of 0, 1 is denoted by ${}^{<\omega}2$. For $f \in {}^{A}B$ we write $A = \mathscr{D}(f)$. ${}^{A}B$ can be considered as a power of the topological space $B$ with the discrete topology. The base for the topology in ${}^{A}B$ is formed by sets $u(\varphi) = \{f \in {}^{A}B;\ f \supseteq \varphi\}$ for $\mathscr{D}(\varphi) \subseteq A$, $\operatorname{card} \mathscr{D}(\varphi) < \omega$. The field of all Borel sets in ${}^{A}B$ is denoted by $\mathscr{B}({}^{A}B)$.

The *complete product* of Boolean algebras $\mathscr{A}_0$, $\mathscr{A}_1$ we define as a triple $(i_0, i_1, \mathscr{B})$, where

(a) $\mathscr{B}$ is a complete Boolean algebra,

(b) $i_0$, $i_1$ are complete injections of $\mathscr{A}_0$, $\mathscr{A}_1$, respectively, into $\mathscr{B}$,

(c) $i_0(\mathscr{A}_0) \cup i_1(\mathscr{A}_1)$ completely generates $\mathscr{B}$.

The *complete independent* product is a complete product satisfying, in addition,

($\mathrm{d}_{\mathrm{ind}}$) $i_0(\mathscr{A}_0)$, $i_1(\mathscr{A}_1)$ are independent in $\mathscr{B}$.

Note. We often consider $\mathscr{A}_0$, $\mathscr{A}_1$ to be regular subalgebras of $\mathscr{B}$ and $i_0$, $i_1$ to be their identity injections. Then we say simply that $\mathscr{B}$ is a complete (independent) product of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$.

For given algebras $\mathscr{A}_0$, $\mathscr{A}_1$, the complete product $(i_0, i_1, \mathscr{B})$ need not be determined uniquely. The same is valid for complete independent products. If $(i'_0, i'_1, \mathscr{B}')$ is another complete product of $\mathscr{A}_0$, $\mathscr{A}_1$, we say that $(i'_0, i'_1, \mathscr{B}') \leqslant (i_0, i_1, \mathscr{B})$ holds, if there is a complete homomorphism $h\colon \mathscr{B} \to \mathscr{B}'$ such that $i'_0 = hi_0$, $i'_1 = hi$. If, moreover, $h$ is an isomorphism, we say that the products $(i_0, i_1, \mathscr{B})$, $(i'_0, i'_1, \mathscr{B}')$ are *isomorphic*.

The above definitions concerning complete products are quite analogous to the well-known definitions of $\mathfrak{m}$-complete independent products (shortly: $(\mathfrak{m}, 0)$-products), where $\mathfrak{m}$ is an infinite cardinal number. R. Sikorski in [7] describes the structure of the set $P_{\mathfrak{m}}$ of all $(\mathfrak{m}, 0)$-products of given Boolean algebras. In $P_{\mathfrak{m}}$ always there exist the so-called minimal product and maximal product. The *minimal product* is a minimal element in $P_{\mathfrak{m}}$. In this paper we show that the minimal product need not be the least one in $P_{\mathfrak{m}}$ (Theorem 2.6). The *maximal product* is the greatest element in $P_{\mathfrak{m}}$.

Let us compare $P_{\mathfrak{m}}$ with the class $P_{\infty}$ of all complete independent products for given $\mathscr{A}_0$, $\mathscr{A}_1$. It is shown in [4] that, when the algebras $\mathscr{A}_0$, $\mathscr{A}_1$ are isomorphic to the Cantor algebra $\mathscr{C}$, then there are complete independent products of arbitrary big cardinality. Therefore, the maximal complete independent product of $\mathscr{A}_0$, $\mathscr{A}_1$ – the greatest element in $P_{\infty}$ – does not exist and $P_{\infty}$ is a proper class. The same is valid, if $\mathscr{A}_0$, $\mathscr{A}_1$ are isomorphic to the random algebra $\mathscr{R}$.

In [7] a bijective correspondence between $(\mathscr{I}, \mathscr{M}, \mathfrak{m})$-extensions of ordinary (i.e. finitely complete) product $\mathscr{A}$ of algebras $\mathscr{A}_0$, $\mathscr{A}_1$ and $(\mathfrak{m}, 0)$-products of $\mathscr{A}_0$, $\mathscr{A}_1$ is described. This correspondence remains valid also for the complete case (the notion of $(\mathscr{I}, \mathscr{M})$-complete extension being understood as the natural extrapolation of the notion of $(\mathscr{I}, \mathscr{M}, \mathfrak{m})$-extension) because any complete independent product $\mathscr{B}$ can be considered as an $(\mathfrak{m}, 0)$-product for some cardinal $\mathfrak{m}$ (it suffices to take $\mathfrak{m}$ such that $\mathscr{B}$ satisfies the $\mathfrak{m}$-chain condition).

By definition, the *minimal complete independent product* (or simply: the minimal product) of algebras $\mathscr{A}_0$, $\mathscr{A}_1$ is the completion of the ordinary Boolean product $\mathscr{A}$ of $\mathscr{A}_0$, $\mathscr{A}_1$. Considering $\mathscr{B}$ as a $(\mathfrak{m}, 0)$-product for proper $\mathfrak{m}$, we get by [7] that the minimal product is the only such product in $P_{\infty}$, or in $P_{\mathfrak{m}}$, that $\mathscr{A}$ is a dense subalgebra of $\mathscr{B}$. Further, the minimal product is a minimal element in $P_{\infty}$, but not necessarily the least one, equally as it is in $P_{\mathfrak{m}}$.

If the algebras $\mathscr{A}_0$, $\mathscr{A}_1$ are atomic, and complete, then they are isomorphic to the fields $\mathscr{P}(X_0)$, $\mathscr{P}(X_1)$ of all subsets of some sets $X_0$, $X_1$ and any complete Boolean product $\mathscr{B}$ of $\mathscr{A}_0$, $\mathscr{A}_1$ is atomic with the set of atoms $X_0 \times X_1$. Thus $\mathscr{B}$ is the only element in $P_{\infty}$. The analogous statement does not hold for infinitely many factors. The counter-example is given by the so-called collapse algebras, which all have countably many complete generators ([8]).

More interesting is the situation when $\mathscr{A}_0$, $\mathscr{A}_1$ are not atomic algebras. However, it is easy to see that, if there are some atoms in $\mathscr{A}_0$, or in $\mathscr{A}_1$ and if $\mathscr{A}_0$, $\mathscr{A}_1$ are complete, then $\mathscr{A}_0$, $\mathscr{A}_1$ or both of them can be decomposed into the direct sum of its, or theirs, non-atomic and atomic parts and any complete independent product of $\mathscr{A}_0$, $\mathscr{A}_1$ can be considered as a direct sum of some products of these parts. The only direct summand which is not determined uniquely is the one constructed from the non-atomic parts. Therefore, it seems reasonable to investigate the extent of $P_{\infty}$ for non-atomic factors. Mostly we shall use the algebras $\mathscr{C}$ or $\mathscr{R}$.

**1. Local properties.** The idea of localization in Boolean products was used, as we have mentioned above, by L. Bukovský in [2], when he was studying the cogeneric extensions. The cogeneric extensions are (two or more) Boolean-valued models with a common generic ultrafilter. As the properties of a generic extensions are determined by the elements belonging to the generic ultrafilter only, so in order to obtain an adequate characterization of cogeneric extensions, dense subsets of Boolean algebras must be investigated. The main tools will be the following definition schemes.

Let $X$ be a property of two subalgebras in a Boolean algebra. We shall say that subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ of an algebra $\mathscr{B}$ are locally $X$ in $\mathscr{B}$, if the set $\mathscr{L}_X = \{U \in \mathscr{B};\ \mathscr{A}_0|U,\ \mathscr{A}_1|U \text{ are } X \text{ in } \mathscr{B}|U\}$ is dense in $\mathscr{B}$, i.e. if for any $W \in \mathscr{B}$, $W \neq 0$ there is $U \in \mathscr{L}_X$, $U \leqslant W$, $U \neq 0$.

Substituting the notion "independent" for $X$ we get the definition of the *local independence*. From the property "disjoint" (subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ are *disjoint* if they have no common elements but zero and unit) we get the *local disjointness*.

The triple $(i_0, i_1, \mathscr{B})$ we call a *complete $X$ product* of algebras $\mathscr{A}_0$, $\mathscr{A}_1$ if it is a complete product of $\mathscr{A}_0$, $\mathscr{A}_1$ and

($d_X$) $i_0(\mathscr{A}_0)$, $i_1(\mathscr{A}_1)$ are $X$ in $\mathscr{B}$.

This scheme involves the definitions of the complete independent, the complete disjoint, the complete locally independent and the complete locally disjoint product. Further, we shall call subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ of $\mathscr{B}$ *nowhere independent*, or *nowhere disjoint*, if for any $V \in \mathscr{B}$, $V \neq 0$ the algebras $\mathscr{A}_0|V$, $\mathscr{A}_1|V$ are not independent, or not disjoint, respectively, in $\mathscr{B}|V$. These notions give other types of complete products.

The investigation of local properties we begin with a general principle of localization. For a subset $x$ of a Boolean algebra $\mathscr{B}$ we denote $x' = \{Y \in \mathscr{B};\ (\forall V \leqslant Y)\, V \neq 0 \rightarrow V \notin x\}$.

THEOREM 1.1. *If $\mathscr{B}$ is a complete Boolean algebra and $x \subseteq \mathscr{B}$, then there is a uniquely determined element $X \in \mathscr{B}$ such that $x$ is dense in $\mathscr{B}|X$ and $x'$ is dense in $\mathscr{B}|-X$.*

Proof. It is easy to verify the following propositions:

(i) $x \cup x'$ is dense in $\mathscr{B}$, $x \cap x' \subseteq \{0\}$,

(ii) $x'$ is convex downwards in $\mathscr{B}$,

(iii) if $x \subseteq y \subseteq \mathscr{B}$, then $x' \supseteq y'$,

(iv) if $x$ is convex downwards in $\mathscr{B}$, then $x'$ is equal to the orthogonal complement $x^{\perp} = \{Y \in \mathscr{B}: (\forall X \in x)\, X \wedge Y = 0\}$,

(v) for $x$ convex downwards, the transformation $x \rightarrow \bar{x} = x''$ is a hull operation, i.e. $x \subseteq \bar{x} = \bar{\bar{x}}$ holds,

(vi) $x$ is dense in $\mathscr{B}|\bigvee \bar{x}$,

(vii) $\bigvee \bar{x} = -\bigvee \bar{x}'$.

The theorem follows from (vi), (vii), if we set $X = \bigvee \bar{x}$. □

From the principle of localization we can derive localization theorems on complete products such as

THEOREM 1.2. *If $\mathscr{B}$ is a complete product of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ then there is $X \in \mathscr{B}$ such that:*

*$\mathscr{B}|X$ is a complete locally independent product of subalgebras $\mathscr{A}_0|X$, $\mathscr{A}_1|X$;*

*$\mathscr{B}|-X$ is a complete locally nowhere independent product of subalgebras $\mathscr{A}_0|-X$, $\mathscr{A}_1|-X$.*

Proof. It suffices to take $x = \mathscr{L}_X$, where $X$ means "independent". □

For independent products we get additional information.

THEOREM 1.3. *Let $\mathscr{B}$ be a complete independent product of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ and let $X$ have the same sense as in Theorem 1.2. Then at least one of the following four possibilities takes place:*

*$\mathscr{A}_0|X$, $\mathscr{A}_0|-X$ are isomorphic to $\mathscr{A}_0$,*

*$\mathscr{A}_1|X$, $\mathscr{A}_1|-X$ are isomorphic to $\mathscr{A}_1$,*

*$\mathscr{A}_0|X$ is isomorphic to $\mathscr{A}_0$ and $\mathscr{A}_1|X$ is isomorphic to $\mathscr{A}_1$,*

*$\mathscr{A}_0|-X$ is isomorphic to $\mathscr{A}_0$ and $\mathscr{A}_1|-X$ is isomorphic to $\mathscr{A}_1$.*

Proof. Let us denote $X_k = h^{\mathscr{A}_k}(X)$, $Y_k = h^{\mathscr{A}_k}(-X)$ for $k = 0, 1$. Then we have $X_k \vee Y_k = 1$ and

$$(X_0 - Y_0) \wedge (Y_1 - X_1) \leqslant -Y_0 \wedge -X_1 \leqslant X \wedge -X = 0.$$

Using the independence we get $X_0 - Y_0 = 0$ or $Y_1 - X_1 = 0$, i.e. $X_0 \leqslant Y_0$ or $Y_1 \leqslant X_1$. That gives $Y_0 = 1$ or $X_1 = 1$, analogously we can derive that $X_0 = 1$ or $Y_1 = 1$ holds. By combination we get the four cases of the theorem (evidently $X_k = 1$ implies that $\mathscr{A}_k|X$ is isomorphic to $\mathscr{A}_k$, analogously for $Y_k = 1$). □

The construction accomplished in Theorem 1.1 is similar to the construction of polars in partially ordered sets described by F. Šik in [10]. A construction similar to Theorem 1.2 is implicitly involved in the proof of Theorems 1 and 2 in [2]. By further analysis of these proofs we can derive the following result.

THEOREM 1.4. *Let $\mathfrak{n}$ be an infinite cardinal number. If $\mathscr{A}_0$, $\mathscr{A}_1$ are complete Boolean algebras, $\mathscr{A}_0$ being $\mathfrak{n}$-distributive and $\mathscr{A}_1$ with a dense subset $\mathscr{M}$ of cardinality $\leqslant \mathfrak{n}$, then any complete product $(i_0, i_1, \mathscr{B})$ of $\mathscr{A}_0$, $\mathscr{A}_1$ is locally independent.*

Proof. Let $(i_0, i_1, \mathscr{B})$ be a complete product of algebras $\mathscr{A}_0$, $\mathscr{A}_1$ satisfying the assumptions of the theorem. By Theorem 1.2, there is an element $X \in \mathscr{B}$ such that $\mathscr{B}|X$ is a complete locally independent product of subalgebras $i_0(\mathscr{A}_0)|X$, $i_1(\mathscr{A}_1)|X$ and $\mathscr{B}|-X$ is a complete locally nowhere independent product of subalgebras $i_0(\mathscr{A}_0)|-X$, $i_1(\mathscr{A}_1)|-X$. We accomplish the proof by showing that $X = 1$ holds.

Proceeding by contradiction we shall assume that the element $-X$ is not zero. Then there is $U \in \mathscr{B}$, $0 \neq U \leqslant -X$ such that $i_0(\mathscr{A}_0)|U$, $i_1(\mathscr{A}_1)|U$

are nowhere independent in $\mathscr{B}|U$. The element $U_0 = h^{i_0(\mathscr{A}_0)}(U)$ is not zero as well and we have $U \wedge i_0(A) \neq 0$ for any $A \in \mathscr{A}_0$, $0 \neq A \leqslant U_0$.

We define a function $f\colon \mathscr{A}_1 \to \mathscr{A}_0$ by setting

$$f(A) = \bigvee \{A_0 \in \mathscr{A}_0;\ i_0(A_0) \wedge i_1(A) \wedge U = 0\}$$

for any $A \in \mathscr{A}_1$. Then we have in $\mathscr{A}_0$

$$\bigwedge \{(f(A) \wedge U_0) \vee (-f(A) \wedge U_0);\ A \in \mathscr{M}\} = U_0$$

and, by n-distributivity of $\mathscr{A}_0$, we get a function $\varepsilon \in {}^{\mathscr{M}}2$ such that

$$\bar{A} = \bigwedge \{(-1)^{\varepsilon(A)} f(A) \wedge U_0;\ A \in \mathscr{M}\} \neq 0.$$

As $\bar{A}$ is an element of $\mathscr{A}_0$ fulfilling $0 \neq \bar{A} \leqslant U_0$, we have $V = i_0(\bar{A}) \wedge U \neq 0$. By assumption, $i_0(\mathscr{A}_0)|U$, $i_1(\mathscr{A}_1)|U$ are nowhere independent in $\mathscr{B}|U$, so $i_0(\mathscr{A}_0)|V$, $i_1(\mathscr{A}_1)|V$ must be dependent. Thus, there exist elements $A_0 \in \mathscr{A}_0$, $A_1 \in \mathscr{A}_1$ such that $i_0(A_0) \wedge V \neq 0$, $i_1(A_1) \wedge V \neq 0$, $i_0(A_0) \wedge i_1(A_1) \wedge V = 0$. Without loss of generality we may assume that $A_0 \leqslant \bar{A}$ and, in view of density of $\mathscr{M}$, $A_1 \in \mathscr{M}$.

We complete the proof by considering the following two cases:

I. $\varepsilon(A_1) = 0$, i.e. $\bar{A} \leqslant f(A_1) \wedge U_0$. Then we have

$$\begin{aligned} i_1(A_1) \wedge V &\leqslant i_1(A_1) \wedge i_0(\bar{A}) \wedge U \leqslant i_1(A_1) \wedge i_0(f(A_1)) \wedge i_0(U_0) \wedge U \\ &= i_1(A_1) \wedge i_0(f(A_1)) \wedge U = 0, \end{aligned}$$

which contradicts $i_1(A_1) \wedge V \neq 0$.

II. $\varepsilon(A_1) = 1$, i.e. $\bar{A} \leqslant -f(A_1) \wedge U_0$. We have then

$$i_0(A_0) \wedge i_1(A_1) \wedge U = i_0(A_0) \wedge i_1(A_1) \wedge V = 0.$$

Therefore, by definition of $f$, $A_0 \leqslant f(A_1)$ holds, which is a contradiction with $0 \neq A_0 \leqslant \bar{A} \leqslant -f(A_1)$. □

Complete locally independent products are closely connected with minimal products. From theorems describing this connection the basic one is

THEOREM 1.5. *If $\mathscr{B}$ is a complete locally independent product of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$, then for any $0 \neq U \in \mathscr{B}$ such that $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$, the algebra $\mathscr{B}|U$ is the minimal product of subalgebras $\mathscr{A}_0|U$, $\mathscr{A}_1|U$.*

Proof. We define

$$\bar{\mathscr{B}} = \{A_0 \wedge A_1;\ A_0 \in \mathscr{A}_0, A_1 \in \mathscr{A}_1\}, \qquad \bar{\bar{\mathscr{B}}} = \{\bigwedge \mathscr{A}, \bigvee \mathscr{A};\ \mathscr{A} \subseteq \bar{\mathscr{B}}\}$$

and prove the following claim.

Claim. $\bar{\mathscr{B}}$ is dense in $\bar{\bar{\mathscr{B}}}$.

It follows from the claim that any element $X \in \bar{\bar{\mathscr{B}}}$ is of the form $X = \bigvee \mathscr{X}$ for some $\mathscr{X} \subseteq \bar{\mathscr{B}}$. Thus, $\bar{\bar{\mathscr{B}}}$ is a complete subalgebra of $\mathscr{B}$, containing

$\mathscr{A}_0 \cup \mathscr{A}_1$, therefore $\bar{\mathscr{B}} = \mathscr{B}$ holds. For any $0 \neq U \in \mathscr{B}$ the set $\{U \wedge X;\ X \in \bar{\mathscr{B}}\}$ is dense in $\mathscr{B}|U$. If, moreover, $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$, then $\mathscr{B}|U$ is the minimal product of subalgebras $\mathscr{A}_0|U$, $\mathscr{A}_1|U$. In this way the claim implies the assertion of the Theorem.

To prove the claim we show that

(*) for any $\mathscr{A} \subseteq \bar{\mathscr{B}}$, such that $\bigwedge \mathscr{A} \neq 0$, there is $X \in \bar{\mathscr{B}}$, $0 \neq X \leqslant \bigwedge \mathscr{A}$.

Let us assume that $\mathscr{A}$ is a subset of $\bar{\mathscr{B}}$ with non-empty intersection. From the local independence of $\mathscr{A}_0$, $\mathscr{A}_1$ in $\mathscr{B}$ we have an element $0 \neq U \leqslant \bigwedge \mathscr{A}$ such that $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$. We denote

$$\mathscr{X}_0 = \{A \in \mathscr{A}_0;\ A \wedge U = 0\}, \qquad \mathscr{X}_1 = \{A \in \mathscr{A}_1;\ A \wedge U = 0\}.$$

For any $A \in \mathscr{X}_0$ we have $A \leqslant -U < 1$, therefore $\bigvee^{\mathscr{A}_0} \mathscr{X}_0 = \bigvee^{\mathscr{B}} \mathscr{X}_0 \neq 1$. So there is $0 \neq X_0 \in \mathscr{A}_0$ such that $X_0 \leqslant -A$ holds for any $A \in \mathscr{X}_0$. Then, for any $A_0 \in \mathscr{A}_0$, the assumption $A_0 \wedge X_0 \neq 0$ implies $A_0 \notin \mathscr{X}_0$, i.e. $A_0 \wedge U \neq 0$. Analogously, there is $0 \neq X_1 \in \mathscr{A}_1$ such that for any $A_1 \in \mathscr{A}_1$ the assumption $A_1 \wedge X_1 \neq 0$ gives $A_1 \notin \mathscr{X}_1$, i.e. $A_1 \wedge U \neq 0$. Using these properties of $X_0$, $X_1$ and the independence of $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ in $\mathscr{B}|U$, we get for any $A_0 \in \mathscr{A}_0$, $A_1 \in \mathscr{A}_1$ that the assumption $A_0 \wedge X_0 \neq 0$, $A_1 \wedge X_1 \neq 0$ implies $A_1 \wedge A_1 \wedge U \neq 0$ and clearly, also $A_0 \wedge A_1 \wedge B \neq 0$ for any $B \in \mathscr{A}$ (it is $U \leqslant \bigwedge \mathscr{A} \leqslant B$).

Now, we are prepared to prove that $X = X_0 \wedge X_1 \leqslant \bigwedge \mathscr{A}$ holds true. Proceeding by contradiction, we assume $-B \wedge X_0 \wedge X_1 \neq 0$ for some $B \in \mathscr{A}$. As $B$ belongs to $\bar{\mathscr{B}}$, there must exist $0 \neq A_0 \in \mathscr{A}_0$, $0 \neq A_1 \in \mathscr{A}_1$ such that $A_0 \wedge A_1 \leqslant -B \wedge X_0 \wedge X_1$. Then we have $A_0 \wedge X_0 \neq 0$, $A_1 \wedge X_1 \neq 0$ but $A_0 \wedge A_1 \wedge B = 0$. This contradiction proves the proposition (*). □

In view of Theorem 1.5 a question arises, which of the elements $U$, in a complete locally independent product $\mathscr{B}$ of algebras $\mathscr{A}_0$, $\mathscr{A}_1$, have the property that $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$. The answer is given by

THEOREM 1.6. *If $\mathscr{B}$ is a complete locally independent product of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$, then any element $U \in \mathscr{B}$ such that $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$ is of the form $U = X_0 \wedge X_1$ for some $X_0 \in \mathscr{A}_0$, $X_1 \in \mathscr{A}_1$.*

Proof. Let us suppose that $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$. Let $\bar{\mathscr{B}}$, $\mathscr{X}_0$, $\mathscr{X}_1$ have the same meaning as in the proof of the previous theorem. We denote $X_0 = -\bigvee \mathscr{X}_0$, $X_1 = -\bigvee \mathscr{X}_1$. Again we have $A_0 \wedge A_1 \wedge U \neq 0$ for any $A_0 \in \mathscr{A}_0$, $A_1 \in \mathscr{A}_1$ such that $A_0 \wedge X_0 \neq 0$, $A_1 \wedge X_1 \neq 0$. By the claim of the previous theorem, $\bar{\mathscr{B}}$ is dense in $\bar{\bar{\mathscr{B}}} = \mathscr{B}$. It implies that $-U \wedge X_0 \wedge X_1 = 0$, i.e. $X_0 \wedge X_1 \leqslant U$. On the other hand, we have $U \wedge \bigvee \mathscr{X}_0 = 0$, $U \wedge \bigvee \mathscr{X}_1 = 0$, i.e. $U \leqslant X_0$, $U \leqslant X_1$, $U \leqslant X_0 \wedge X_1$. □

Using the notion of the localized minimality (a complete product $\mathscr{B}$ of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ is *locally minimal* if the set $\{U \in \mathscr{B};\ \mathscr{B}|U$ is the minimal product of $\mathscr{A}_0|U,\ \mathscr{A}_1|U\}$ is dense in $\mathscr{B}$) we can formulate Theorem 1.5 in the following way.

THEOREM 1.7. *Any complete locally independent product is locally minimal.*

Combining Theorems 1.2 and 1.7 we get

THEOREM 1.8. *If $\mathscr{B}$ is a complete product of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$, then there is $X \in \mathscr{B}$ such that*

*$\mathscr{B}|X$ is a locally minimal product of $\mathscr{A}_0|X$, $\mathscr{A}_1|X$;*

*$\mathscr{B}|-X$ is a complete locally nowhere independent product of $\mathscr{A}_0|-X$, $\mathscr{A}_1|-X$.*

Under assumption of independence, Theorems 1.5, 1.6 can be expressed in a stronger form.

THEOREM 1.9. *A complete independent product $\mathscr{B}$ of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ is locally independent if and only if it is the minimal product of $\mathscr{A}_0$, $\mathscr{A}_1$. In that case, for any $U \in \mathscr{B}$, the algebras $\mathscr{A}_0|U$, $\mathscr{A}_1|U$ are independent in $\mathscr{B}|U$ if and only if $U$ is of the form $U = X_0 \wedge X_1$ for some $X_0 \in \mathscr{A}_0$, $X_1 \in \mathscr{A}_1$.*

Remark. The first part of Theorem 1.9 was found by analyzing one result on cogeneric extensions proved by L. Bukovský (see [3]) using the method of Boolean-valued models and generic ultrafilters.

Theorems 1.4, 1.9 give as a consequence

THEOREM 1.10. *Let $\mathfrak{n}$ be an infinite cardinal number. If $\mathscr{A}_0$, $\mathscr{A}_1$ are complete Boolean algebras, $\mathscr{A}_0$ being $\mathfrak{n}$-distributive and $\mathscr{A}_1$ with a dense subset $\mathscr{M}$ of cardinality $\leqslant \mathfrak{n}$, then there is exactly one complete independent product of $\mathscr{A}_0$, $\mathscr{A}_1$, the minimal one.*

In the rest of this section we discuss the relations between the notions of independence, disjointness, local independence and local disjointness of complete products. It is easy to see that independent subalgebras must be disjoint. The same implication is valid for localized notions.

We shall show that except these two trivial implications no other implication between the four notions takes place. It is a consequence of the following four theorems.

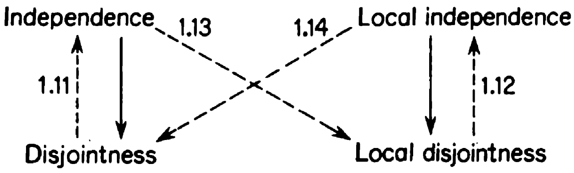
THEOREM 1.11. *There exists a complete disjoint product which is not independent.*

THEOREM 1.12. *There exists a complete locally disjoint product which is not locally. independent.*

THEOREM 1.13. *There exists a complete independent product which is not locally disjoint.*

THEOREM 1.14. *There exists a complete locally independent product which is not disjoint.*

The situation is described by the following diagram (the full arrows correspond to valid implications, the interrupted arrows to false ones).




The implications corresponding to the arrows which are not drawn on the diagram, are false. For example the independence does not imply the local independence, because it would be in contradiction with 1.13. By the same reason, the disjointness does not imply the local disjointness. Theorem 1.14 has analogous consequences in the converse direction.

Proof of Theorem 1.11. It suffices to take any complete independent product $\mathscr{B}$ of subalgebras $\mathscr{A}_0$, $\mathscr{A}_1$ which are not two-elemented. We take elements $A_0 \in \mathscr{A}_0$, $A_1 \in \mathscr{A}_1$ different from 0 and 1 and set $A = -(A_0 \wedge A_1)$. Then $\mathscr{B}|A$ is a complete disjoint product of $\mathscr{A}_0|A$, $\mathscr{A}_1|A$ which in addition are isomorphic with $\mathscr{A}_0$, $\mathscr{A}_1$, respectively. □

The proofs of Theorems 1.12, 1.13 and 1.14 we give in the next section where we present various examples of complete products for algebras $\mathscr{C}$, $\mathscr{R}$.

**2. Products of algebras $\mathscr{C}$ and $\mathscr{R}$.** The first section was devoted to investigation of properties of complete Boolean products. With exception of the minimal product, we left open the question of existence of complete products. This question will be considered now. We bring several complete Boolean products using the algebras $\mathscr{C}$ and $\mathscr{R}$ as factors and investigate their properties. The method of construction is based on the following theorem of rather technical character.

THEOREM 2.1. *Let be $f \in {}^{\omega}\omega$, then for any $\varphi \in {}^{<\omega}2$ we define the function $\varphi_f$ by setting $\varphi_f(f(k)) = \varphi(k)$ for any $k \in \mathscr{D}(\varphi)$. If $f$ is an injection, then there is exactly one complete injection $i$: $\mathscr{C} \to \mathscr{C}$ such that for any $\varphi \in {}^{<\omega}2$ it is $i([u(\varphi)]_{\mathscr{C}}) = [u(\varphi_f)]_{\mathscr{C}}$. Moreover, if we denote $X = [u(\psi)]_{\mathscr{C}}$ for some $\psi \in {}^{<\omega}2$, $\mathscr{D}(\psi) \cap \mathscr{W}(f) \neq 0$, and if $i_X$: $\mathscr{C} \to \mathscr{C}|X$ denotes the natural homomorphism $A \mapsto A \wedge X$, then $i_X i$: $\mathscr{C} \to \mathscr{C}|X$ is also a complete injection. The proposition remains valid, when the algebra $\mathscr{C}$ is replaced by the algebra $\mathscr{R}$.*

Proof. We define a mapping $\bar{i}$: $\mathscr{P}({}^{\omega}2) \to \mathscr{P}({}^{\omega}2)$ as follows: $\bar{i}(A) = \{g \in {}^{\omega}2;\ gf \in A\}$ for any $A \subseteq {}^{\omega}2$. It is easy to verify that $\bar{i}$ preserves set unions and intersections and that $\bar{i}$ also preserves properties of sets to be open, borel, or of zero measure. Further, for any $\varphi \in {}^{<\omega}2$ we have $i(u(\varphi))$

$= u(\varphi_f)$, thus by factorization of $\bar{i}|\mathscr{B}({}^{\omega}2)$ by the ideal $\mathscr{I}_{\mathscr{C}}$ or $\mathscr{I}_{\mathscr{R}}$ we get the desired injection $i$: $\mathscr{C} \to \mathscr{C}$, or $i$: $\mathscr{R} \to \mathscr{R}$, respectively. It is clear that in both cases the injection is complete and uniquely defined.

To prove the second part of the theorem, it suffice to show that, in denotation $X = [u(\psi)]_{\mathscr{C}}$, or $X = [u(\psi)]_{\mathscr{R}}$, for $\psi \in {}^n2$, $n \in \omega$, $\mathscr{D}(\psi) \cap \mathscr{W}(f) = \emptyset$, the mapping $i_X$: $\mathscr{C} \to \mathscr{C}|X$, or $i_X$: $\mathscr{R} \to \mathscr{R}|X$, respectively, is injective. For $\mathscr{C}$ it is a consequence of the fact that the set $\{[u(\varphi)]_{\mathscr{C}};\ \varphi \in {}^{<\omega}2\}$ is a base in $\mathscr{C}$ and that for any $\varphi \in {}^{<\omega}2$, $\mathscr{D}(\varphi_f) \cap \mathscr{D}(\psi) = \emptyset$, i.e. $\bar{i}(u(\varphi)) \cap u(\psi) \neq 0$ holds. For $\mathscr{R}$ we have the measure $\mu$:

$$\mu\big(\bar{i}(u(\varphi)) \cap u(\psi)\big) = 2^{-n} \cdot \mu\big(u(\varphi)\big),$$

thus for $A \in \mathscr{B}({}^{\omega}2)$ it is $\mu\big(\bar{i}(A) \cap u(\psi)\big) = 2^{-n} \cdot \mu(A)$. □

The main application of Theorem 2.1 will be in defining a special complete product $(i_0^{\mathscr{C}}, i_1^{\mathscr{C}}, \mathscr{C})$ of the algebras $\mathscr{C}$, $\mathscr{C}$, or $(i_0^{\mathscr{R}}, i_1^{\mathscr{R}}, \mathscr{R})$ of $\mathscr{R}$, $\mathscr{R}$, respectively, which we shall call the quadratic product. Roughly speaking, the quadratic product of $\mathscr{C}$, $\mathscr{C}$ or $\mathscr{R}$, $\mathscr{R}$, is the algebra of all borel subsets of the unit square in the real plane factorized by the corresponding ideal of all meager sets, or sets of zero measure, respectively. This is not said very exactly, because the mentioned algebra is isomorphic to $\mathscr{C}$, or to $\mathscr{R}$, so the essential point is how the factors are embedded into it. Geometrically, the embeddings can be described as the maximal inverse mapping to the orthogonal projections of the square onto its edges. In the arithmetical language, we represent each point by a sequence of 0's and 1's and then we use even members of the sequence for the first projection and odd members for the second one. The detailed definition sounds as follows.

We define the functions $f_0$, $f_1 \in {}^{\omega}\omega$ by setting $f_0(k) = 2k$, $f_1(k) = 2k+1$ for any $k \in \omega$. By Theorem 2.1, the functions $f_0$, $f_1$ induce complete embeddings $i_0^{\mathscr{C}}, i_1^{\mathscr{C}}$: $\mathscr{C} \to \mathscr{C}$, $i_0^{\mathscr{R}}, i_1^{\mathscr{R}}$: $\mathscr{R} \to \mathscr{R}$ such that for any $\varphi \in {}^{<\omega}2$, $t \in 2$ it is

$$i_t([u(\varphi)]_{\mathscr{C}}) = [u(\varphi_{f_t})]_{\mathscr{C}}$$

and

$$i_t([u(\varphi)]_{\mathscr{R}}) = [u(\varphi_{f_t})]_{\mathscr{R}}.$$

THEOREM 2.2. *The triple $(i_0^{\mathscr{C}}, i_1^{\mathscr{C}}, \mathscr{C})$ is a complete independent product of the algebras $\mathscr{C}$, $\mathscr{C}$.*

Proof. It suffices to verify that $i_0^{\mathscr{C}}(\mathscr{C})$, $i_1^{\mathscr{C}}(\mathscr{C})$ are independent subalgebras in $\mathscr{C}$ and that their union completely generates $\mathscr{C}$. These two prepositions are consequences of the fact that for given $n \in \omega$, the pairs $(\varphi, \psi) \in {}^n2 \times {}^n2$ correspond in one-to-one manner to the elements $\vartheta \in {}^{2n}2$ by relation $\vartheta = (\varphi(0), \psi(0), \varphi(1), \psi(1), \ldots)$. This relation implies

$$i_0^{\mathscr{C}}([u(\varphi)]_{\mathscr{C}}) \wedge i_1^{\mathscr{C}}([u(\psi)]_{\mathscr{C}}) = [u(\varphi_{f_0})]_{\mathscr{C}} \wedge [u(\psi_{f_1})]_{\mathscr{C}} = [u(\vartheta)]_{\mathscr{C}}.$$

Moreover, we use the fact that the set $\{[u(\vartheta)]_{\mathscr{C}};\ \vartheta\in {}^{<\omega}2\}$ is a base in $\mathscr{C}$. □

THEOREM 2.3. *The triple* $(i_0^{\mathscr{R}}, i_1^{\mathscr{R}}, \mathscr{R})$ *is a complete independent product of the algebras* $\mathscr{R}$, $\mathscr{R}$.

Proof. The proof is similar to the previous one, with the exception that the set $\{[u(\vartheta)]_{\mathscr{R}};\ \vartheta\in {}^{<\omega}2\}$ is not a base in $\mathscr{R}$. We shall find another base. We define $\Omega$ as the set of all unions of the form $\Phi = \bigcup\{\Phi_n;\ n\in\omega\}$ where it is $\Phi_n \subseteq {}^n2$ for any $n\in\omega$ and where $\varphi\in\Phi_n$ implies $\varphi|k\in\Phi_k$ for any $k < n\in\omega$. For $\Phi\in\Omega$, $t\in 2$ we denote

$$\mathscr{U}(\Phi) = \bigcap\{\bigcup\{u(\varphi);\ \varphi\in\Phi_n\};\ n\in\omega\},$$
$$\Phi_{f_t} = \{\varphi_{f_t};\ \varphi\in\Phi\}.$$

The weak $\omega$-distributivity of $\mathscr{R}$ implies that the set $\{[\mathscr{U}(\Phi)]_{\mathscr{R}};\ \Phi\in\Omega\}$ is a base in $\mathscr{R}$. There $[\mathscr{U}(\Phi)]_{\mathscr{R}} \neq 0$ holds if and only if $\mu(\mathscr{U}(\Phi)) > 0$, i.e. if

$$\lim_{n\in\omega} \operatorname{card}\Phi_n\cdot 2^{-n} > 0.$$

By an easy calculation we get for $\Phi$, $\Psi\in\Omega$, $\mu(\mathscr{U}(\Phi)) > 0$, $\mu(\mathscr{U}(\Psi)) > 0$ that

$$\mu(\mathscr{U}(\Phi_{f_0}) \cap \mathscr{U}(\Psi_{f_1})) = \lim_{n\in\omega} \operatorname{card}\Phi_n\cdot\operatorname{card}\Psi_n\cdot 2^{-2n} > 0$$

holds. Therefore, the subalgebras $i_0^{\mathscr{R}}(\mathscr{R})$, $i_1^{\mathscr{R}}(\mathscr{R})$ are independent in $\mathscr{R}$. The complete genericity is proved in the same way as in Theorem 2.2. □

The just described quadratic products $(i_0^{\mathscr{C}}, i_1^{\mathscr{C}}, \mathscr{C})$ and $(i_0^{\mathscr{R}}, i_1^{\mathscr{R}}, \mathscr{R})$ we shall investigate from the point of view of the local independence and of the local disjointness. We compare the quadratic products with the minimal products of the same algebras. However, before doing so, we show another use of Theorems 2.1 for the construction of a complete locally independent not disjoint product and of a complete independent not locally disjoint product. These examples will provide the proofs of Theorems 1.13 and 1.14.

Proof of Theorem 1.14. We define mappings $f_0$, $f_1\in{}^\omega\omega$ by setting $f_0(k) = 2k$ for any $k\in\omega$ and $f_1(k) = 0$ for $k = 0$, $f_1(k) = 2k-1$ for $k > 0$, $k\in\omega$. The injections $i_0$, $i_1$: $\mathscr{C}\to\mathscr{C}$ are induced by $f_0$, $f_1$ following Theorem 2.1. We show that $(i_0, i_1, \mathscr{C})$ is a complete product of $\mathscr{C}$, $\mathscr{C}$ with the desired properties. The subalgebras $i_0(\mathscr{C})$, $i_1(\mathscr{C})$ are not disjoint in $\mathscr{C}$ because

$$i_0(u((0))) = i_1(u((0))) = ((0))$$

holds. (Symbols (0) and (1) denote one-element sequences with the only element 0, 1 respectively.) To prove the local independence of $i_0(\mathscr{C})$, $i_1(\mathscr{C})$ it suffices to show that for any $\varphi\in{}^n2$, $n > 0$ the subalgebras $i_0(\mathscr{C})|u(\varphi)$, $i_1(\mathscr{C})|u(\varphi)$ are independent in $\mathscr{C}|u(\varphi)$. If $\psi$, $\chi\in{}^{<\omega}2$ are such that $i_0(u(\psi)) \wedge u(\varphi) \neq 0$, $i_1(u(\chi)) \wedge u(\varphi) \neq 0$ holds true, then the functions $\psi_{f_0}$,

$\chi_{f_1}$ are compatible with $\varphi$. That implies $\psi(0) = \chi(0) = \varphi(0)$ and therefore for $\vartheta = (\varphi(0), \chi(1), \psi(1), \ldots)$ we have

$$i_0(u(\psi)) \wedge i_1(u(\chi)) = u(\vartheta) \neq 0.$$

Finally, the subalgebras $i_0(\mathscr{C})$, $i_1(\mathscr{C})$ completely generate $\mathscr{C}$, because for any $\vartheta \in {}^{<\omega}2$ there are $\psi$, $\chi \in {}^{<\omega}2$ such that it is $\vartheta = (\psi(0), \chi(1), \psi(1), \ldots) = (\chi(0), \chi(1), \psi(1), \ldots)$ and therefore

$$u(\vartheta) = i_0(u(\psi)) \wedge i_1(u(\chi)). \ \square$$

Proof of Theorem 1.13 is similar to the previous one. The mappings $f_0, f_1, f \in {}^{\omega}\omega$ are defined by $f_0(k) = 2k+1$, $f_1(k) = 2k+2$, $f(k) = k+1$ for any $k \in \omega$, the injections

$$i_0^*\colon \mathscr{C} \to \mathscr{C}|u((0)), \qquad i_1^*\colon \mathscr{C} \to \mathscr{C}|u((0)), \qquad i^*\colon \mathscr{C} \to \mathscr{C}|u((1))$$

are induced by $f_0$, $f_1$, $f$, respectively, following Theorem 2.1. We define injections $i_0, i_1\colon \mathscr{C} \to \mathscr{C}$ as the direct sums of mappings $i_0 = i_0^* + i^*$, $i_1 = i_1^* + i^*$, i.e. we have

$$i_0(u(\varphi)) = u(\varphi_{f_0}) \vee u(\varphi_f), \qquad i_1(u(\varphi)) = u(\varphi_{f_1}) \vee u(\varphi_f)$$

for any $\varphi \in {}^{<\omega}2$. We show now that $(i_0, i_1, \mathscr{C})$ is a complete product of $\mathscr{C}$, $\mathscr{C}$ which is independent but not locally disjoint.

The local not-disjointness of $i_0(\mathscr{C})$, $i_1(\mathscr{C})$ in $\mathscr{C}$ follows from the fact that the mappings $i_0$, $i_1$ coincide on $u((1))$ (they both are equal to $i^*$ there). The independence of $i_0(\mathscr{C})$, $i_1(\mathscr{C})$ can be verified analogously as the local independence in the previous proof.

To prove that $\mathscr{C}$ is completely generated by $i_0(\mathscr{C})$, $i_1(\mathscr{C})$ we notice first that

$$u((1)) = -u((0)) = \bigwedge \{\bigvee \{i_0(u(\varphi)) \wedge i_1(u(\varphi));\ \varphi \in {}^k2\};\ k \in \omega\}$$

holds. The rest is obvious. $\square$

Now we come to the investigation of the local properties of the quadratic and of the minimal product of algebras $\mathscr{C}$, $\mathscr{C}$ and $\mathscr{R}$, $\mathscr{R}$.

THEOREM 2.4. *The quadratic product* $(i_0^{\mathscr{C}}, i_1^{\mathscr{C}}, \mathscr{C})$ *of algebras* $\mathscr{C}$, $\mathscr{C}$ *is locally independent.*

Proof. The assertion follows from the fact that the set $\{u(\varphi);\ \varphi \in {}^{<\omega}2\}$ is dense in $\mathscr{C}$.

COROLLARY 1. *The quadratic product of algebras* $\mathscr{C}$, $\mathscr{C}$ *is locally disjoint.*

COROLLARY 2. *The quadratic product of algebras* $\mathscr{C}$, $\mathscr{C}$ *is isomorphic with their minimal product.*

Proof. By Theorem 1.9. $\square$

The fact that for algebras $\mathscr{C}$, $\mathscr{C}$ the minimal and the quadratic product are isomorphic, pointed out L. Bukovský in [2]. For algebras $\mathscr{R}$, $\mathscr{R}$ he proved that the mentioned two products are not isomorphic. We strengthen this result in the following way.

THEOREM 2.5. *The quadratic product* $(i_0^{\mathscr{R}}, i_1^{\mathscr{R}}, \mathscr{R})$ *of algebras* $\mathscr{R}$, $\mathscr{R}$ *is incomparable with the minimal product* $(i_0^{\otimes}, i_1^{\otimes}, \mathscr{R}\otimes\mathscr{R})$ *in the class* $P_\infty$ *of all complete independent products of algebras* $\mathscr{R}$, $\mathscr{R}$.

Proof. Proceeding by contradiction, let us assume that, in $P_\infty$, $(i_0^{\otimes}, i_1^{\otimes}, \mathscr{R}\otimes\mathscr{R}) \leqslant (i_0^{\mathscr{R}}, i_1^{\mathscr{R}}, \mathscr{R})$ holds, i.e. that there is a complete homomorphism $h\colon \mathscr{R} \to \mathscr{R}\otimes\mathscr{R}$ such that $i_k^{\otimes} = hi_k^{\mathscr{R}}$ for $k\in 2$.

It is proved in [7], 38.B, that if $N$ is a nowhere dense subset of the real unit interval $I$ and if $\mu(N) > 0$, then the closed set $S = \{(x_0, x_1)\in I\times I;\ |x_0-x_1|\in N\}$ has the property

$(*)$ for any two Borel subsets $A_0, A_1 \subseteq I$, $\mu(A_0\times A_1) > 0$ implies $\mu(A_0\times A_1 - S) > 0$.

Clearly, in $I$, there exists a sequence $(N_n;\ n\in\omega)$ of nowhere dense subsets such that $\lim_{n\in\omega}\mu(N_n) = 1$ holds. Using the above mentioned Sikorski's assertion and the well-known Fubini Theorem, we get in $I\times I$ a sequence of closed subsets $(S_n;\ n\in\omega)$ fulfilling $(*)$ and the condition $\lim_{n\in\omega}\mu(S_n) = 1$. This implies that in $\mathscr{R}$ there is a sequence of elements $(T_n;\ n\in\omega)$ such that $\lim_{n\in\omega}\mu(T_n) = 1$ holds and

$(**)$ for any $n\in\omega$, $A_0, A_1\in\mathscr{R}$, $A_0, A_1 \neq 0$ implies $i_0^{\mathscr{R}}(A_0) \wedge i_1^{\mathscr{R}}(A_1) - T_n \neq 0$.

We denote $X = -\bigvee \operatorname{Ker} h$ and so we get the isomorphism $i\colon \mathscr{R}|X \to \mathscr{R}\otimes\mathscr{R}$ fulfilling $h(A) = i(A\wedge X)$ for any $A\in\mathscr{R}$. Assuming $h(T_n) = i(T_n\wedge X) \neq 0$ for some $n\in\omega$ we get non-zero elements $A_0, A_1\in\mathscr{R}$ such that $i_0^{\mathscr{R}}(A_0)\wedge i_1^{\mathscr{R}}(A_1) \leqslant T_n\wedge X \leqslant T_n$ which is in contradiction with $(**)$. Therefore, it is $h(T_n) = 0$, i.e. $T_n\in\operatorname{Ker} h$ for any $n\in\omega$. Then, by the completeness of $\operatorname{Ker} h$, we have $\bigvee\{T_n;\ n\in\omega\}\in\operatorname{Ker} h$. We have got a contradiction with $\mu(\bigvee\{T_n;\ n\in\omega\}) = \lim_{n\in\omega}\mu(T_n) = 1$. The proof is complete because of the minimality of $(i_0^{\otimes}, i_1^{\otimes}, \mathscr{R}\otimes\mathscr{R})$ in $P_\infty$. □

Theorem 2.5 gives two interesting consequences. The first one is connected with the structure of $P_\infty$ or $P_\omega$ for algebras $\mathscr{R}$, $\mathscr{R}$, the second one concerns the local independence of the quadratic product of $\mathscr{R}$, $\mathscr{R}$.

THEOREM 2.6. *There is no least element in the class* $P_\infty$ (*or* $P_\omega$) *of all complete* (*or* $\omega$*-complete, respectively*) *independent Boolean products of algebras* $\mathscr{R}$, $\mathscr{R}$.

Proof. For $P_\infty$, it follows directly from Theorem 2.5. For $P_\omega$, we notice that the quadratic product of $\mathcal{R}$, $\mathcal{R}$, satisfying the countable chain condition (c.c.c.), contains the ordinary Boolean product $\mathcal{A}$ of $\mathcal{R}$, $\mathcal{R}$, which, in its turn, is contained as a dense subalgebra in the minimal complete product $\mathcal{R}\otimes\mathcal{R}$. Therefore, in $\mathcal{A}$ and in $\mathcal{R}\otimes\mathcal{R}$, the c.c.c. is fulfilled and the minimal product $\mathcal{R}\otimes\mathcal{R}$ belongs to $P_\omega$.

Remark. In fact, both Theorems 2.5, 2.6 are valid in $P_\mathfrak{m}$ for any infinite cardinal $\mathfrak{m}$.

THEOREM 2.7. *The quadratic product of algebras $\mathcal{R}$, $\mathcal{R}$ is not locally independent.*

Proof. By Theorems 1.9, 2.5. □

The last theorem of this section describes the local disjointness of the quadratic product of $\mathcal{R}$, $\mathcal{R}$. At the same time it presents (together with Theorem 2.7) a proof of Theorem 1.12.

THEOREM 2.8. *The quadratic product of algebras $\mathcal{R}$, $\mathcal{R}$ is locally disjoint.*

Proof. Proceeding by contradiction, let us assume that the quadratic product $(i_0^{\mathcal{R}}, i_1^{\mathcal{R}}, \mathcal{R})$ is not locally disjoint. That means that

(*) there is a non-zero element $U \in \mathcal{R}$ such that for any non-zero $V \in \mathcal{R}$, $V \leqslant U$ the algebras $i_0^{\mathcal{R}}(\mathcal{R})|V$, $i_1^{\mathcal{R}}(\mathcal{R})|V$ are not disjoint, i.e. there exist elements $A_0, A_1 \in \mathcal{R}$ such that $0 < i_0^{\mathcal{R}}(A_0) \wedge V = i_1^{\mathcal{R}}(A_1) \wedge V < V$.

The element $U$, being Lebesgue measurable and of positive measure, can be covered in $I \times I$ by disjoint intervals of total measure less than $\frac{4}{3}\mu(U)$. Among them, there exists an interval $I_0 \times I_1$ such that $\mu(I_0 \times I_1) < \frac{4}{3}(U \cap \cap (I_0 \times I_1))$. Without loss of generality we may assume that $I_0 \times I_1 = I \times I$ holds and we have $\mu(U) > \frac{3}{4}$.

It is easy to verify that

$$\mathcal{A} = \{A_0 \in \mathcal{R};\ (\exists A_1 \in \mathcal{R})\, i_0^{\mathcal{R}}(A_0) \wedge U = i_1^{\mathcal{R}}(A_1) \wedge U\}$$

is a complete subalgebra in $\mathcal{R}$. Moreover, for any $A_0 \in \mathcal{A}$ we have $\mu(A_0) < \frac{1}{4}$ or $\mu(A_0) > \frac{3}{4}$. It is a consequence of the fact that

$$\begin{aligned} U &= \big(i_0^{\mathcal{R}}(A_0) \wedge U\big) \vee \big(i_0^{\mathcal{R}}(-A_0) \wedge U\big) \\ &= \big(i_0^{\mathcal{R}}(A_0) \wedge i_1^{\mathcal{R}}(A_1) \wedge U\big) \vee \big(i_0^{\mathcal{R}}(-A_0) \wedge i_1^{\mathcal{R}}(-A_1) \wedge U\big) \end{aligned}$$

and

$$\mu(U) \leqslant \mu(A_0)\cdot\mu(A_1) + \mu(-A_0)\cdot\mu(-A_1).$$

(For $\frac{1}{4} \leqslant \mu(A_0) \leqslant \frac{3}{4}$ we get, by an easy computation, $\mu(U) \leqslant \frac{3}{4}$ for any $A_1$.)

For $A, A' \in \mathcal{I} = \{A \in \mathcal{A};\ \mu(A) < \frac{1}{4}\}$ we have $\mu(A \vee A') \not> \frac{3}{4}$. So $\mathcal{I}$ is an ideal in $\mathcal{A}$. $\mathcal{I}$ preserves unions of increasing sequences and is, therefore, complete. Let $\bar{A}$ be the maximal element in $\mathcal{I}$, then $0 \neq A \in \mathcal{A}$, $A \wedge \bar{A} = 0$

implies $\mu(A) > \frac{3}{4}$. This leads to a contradiction with the assumption (*), because the element $-A$ cannot be divided into two disjoint parts, each of measure $> \frac{3}{4}$. The theorem is proved. □

Remark. The idea of this proof is due to J. Cichoń. The original proof will be published in [5], in a more general situation.

**3. Cohen and random reals.** The localised algebraic properties of complete Boolean products are connected with properties of the corresponding Boolean-valued models, the so-called cogeneric extensions (see [2]). So the products of algebras $\mathscr{C}$ or $\mathscr{R}$ are connected with cogeneric extension by a Cohen or random real, respectively. Properties of Cohen and random reals from this point of view are the subject of investigation in this section.

A real number $r$ is called *Cohen*, or *random*, *over a model class* $M$, if $r$ is contained in any open dense subset, or in any subset of measure 1, respectively, of real unit interval, belonging to $M$. R. Solovay in [9] proved that an extension $M(r)$ of a model class by a real number $r$ is isomorphic with the Boolean model $M^{\mathscr{C}}$, or $M^{\mathscr{R}}$, if and only if $r$ is Cohen, or random, respectively, over $M$. As it is usual, we shall identify real numbers from the unit interval with elements of ${}^{\omega}2$. The sets $u(\varphi)$, $\varphi \in {}^{<\omega}2$ will be referred to as rational intervals.

If $M \subseteq M_0$, $M_1$ are transitive model classes of ZFC, we say that $M_0$, $M_1$ are *separated over* $M$ if for any disjoint sets $R_0, R_1 \subseteq M$, $R_0 \in M_0$, $R_1 \in M_1$ there is a set $S \in M$ such that $R_0 \subseteq S$, $R_1 \cap S = \emptyset$. We say that $M_0$, $M_1$ are *disjoint over* $M$ if $M_0 \cap M_1 = M$.

In the following two theorems we give sufficient conditions for Cohen or random extensions of a model class $M$, not to be separated or disjoint over $M$. Later, the applications of these conditions are shown.

THEOREM 3.1. *Let $M$ be a transitive model of* ZFC, *let each of real numbers $r_0$, $r_1$ be Cohen or random over $M$. If there is a partition $\{N_k;\ k \in \omega\} \in M$ of a subset $N \subseteq \omega$ such that* $\operatorname{card} N_k = n_k < \omega$, $r_0|N_k \neq r_1|N_k$ *holds for every $k \in \omega$, then there exist disjoint subsets $R_0$, $R_1 \subseteq \omega$, $R_0 \in M(r_0)$, $R_1 \in M(r_1)$ which are not separated over $M$* (i.e. *such that there is no $S \in M$, $R_0 \subseteq S$, $R_1 \subseteq \omega - S$*).

Proof. We denote $\mathscr{N}_k = {}^{N_k}2$ for any $k \in \omega$,

$$\mathscr{N} = \bigcup \{\mathscr{N}_k;\ k \in \omega\}, \qquad \bar{R}_0 = \{r_0|N_k;\ k \in \omega\}, \qquad \bar{R}_1 = \{r_1|N_k;\ k \in \omega\}.$$

The sets $\bar{R}_0$, $\bar{R}_1$ are disjoint subsets of the countable set $\mathscr{N} \in M$. By a standard bijection belonging to $M$, can $\bar{R}_0$, $\bar{R}_1$ be mapped to disjoint subsets $R_0, R_1 \subseteq \omega$. Then we have $R_0 \in M(r_0)$, $R_1 \in M(r_1)$ and it is clear that the non-separation of $R_0$, $R_1$ over $M$ is equivalent to the non-separation of $\bar{R}_0$, $\bar{R}_1$ over $M$.

Let us assume that there is $S \in M$ such that $\bar{R}_0 \subseteq S$, $\bar{R}_1 \subseteq \mathcal{N} - S$. We denote $S_k = \mathcal{N}_k \cap S$ for $k \in \omega$ and we get

$$2^{-n_k} \cdot \operatorname{card} S_k + 2^{-n_k} \cdot \operatorname{card}(\mathcal{N}_k - S_k) = 2^{-n_k} \cdot \operatorname{card} \mathcal{N}_k = 1$$

for any $k \in \omega$. Thus, without loss of generality, we may assume that for infinitely many $k \in \omega$ the inequality $2^{-n_k} \cdot \operatorname{card} S_k \leqslant \frac{1}{2}$ holds.

If we denote, for $j \in \omega$,

$$u_j = \bigcap \{\bigcup \{u(\varphi);\ \varphi \in S_k\};\ k \in j\} = \bigcup \{u(\varphi);\ \varphi \in S_0 \times S_1 \times \ldots \times S_{j-1}\},$$

then $U = (u_j;\ j \in \omega) \in M$ is a decreasing sequence of closed subsets in real unit interval and their measures converge to zero. Therefore, $\bigcap U$ is a closed nowhere dense subset of measure zero, belonging to $M$. At the same time $r_0 \in \bigcap U$ holds, which is a contradiction with $r_0$ being Cohen or random over $M$. □

THEOREM 3.2. *Let $M$ be a transitive model of* ZFC, *let each of real numbers $r_0$, $r_1$ be Cohen or random over $M$. If there is an infinite subset $a \subseteq \omega$, $a \in M$ such that $r_0(k) \neq r_1(k)$ holds for every $k \in a$, then there is a real number $r \in M(r_0) \cap M(r_1)$ which is Cohen or random over $M$.*

Proof. We denote $\bar{r}_0 = r_0|a \in M(r_0)$, $\bar{r}_1 = r_1|a \in M(r_1)$. The restriction of Cohen, or random number over $M$ to an infinite set $a \in M$ is Cohen, or random, respectively. Moreover, we have $\bar{r}_0(k) \equiv \bar{r}_1(k) + 1 \pmod 2$ for every $k \in a$, i.e. $\bar{r}_0$ belongs to $M(r_0) \cap M(r_1)$. □

Now, we describe several cases in which the conditions of Theorems 3.1, or 3.2 are fulfilled.

THEOREM 3.3. *Let $M$ be a transitive model of* ZFC. *If $r \in {}^{\omega}2$ is a random real over $M$ and $r_0$, $r_1$ are the restrictions of $r$ to the set of all even and to the set of all odd numbers, respectively, then there exist disjoint subsets $R_0$, $R_1 \subseteq \omega$, $R_0 \in M(r_0)$, $R_1 \in M(r_1)$ which are not separated over $M$.*

Proof. We shall find a partition with the properties described in Theorem 3.1. It is evident that we can choose a sequence $(\{N_{jk};\ k \in \omega\};\ j \in \omega)$ of partitions of $\omega$ in such a way that for any $j \leqslant j' \in \omega$ the partition $\{N_{jk};\ k \in \omega\}$ is a refinement of $\{N_{j'k};\ k \in \omega\}$ and, in denotation $\operatorname{card} N_{jk} = n_{jk} < \omega$, the sum $\sum \{2^{-n_{jk}};\ k \in \omega\} = n_j$ exists and $\lim_{j \in \omega} n_j = 0$ holds.

In the real unit square ${}^{\omega}2 \times {}^{\omega}2$ let us denote

$$u_{jk} = \bigcup \{u(\varphi) \times u(\varphi);\ \varphi \in {}^{N_{jk}}2\}, \qquad u_j = \bigcup \{u_{jk};\ k \in \omega\}, \qquad U = (u_j;\ j \in \omega).$$

Then we have

$$\mu(u_j) \leqslant \sum \{\mu(u_{jk});\ k \in \omega\} = \sum \{2^{-n_{jk}};\ k \in \omega\} = n_j.$$

Thus $U \in M$ is a decreasing sequence of open subsets of ${}^{\omega}2 \times {}^{\omega}2$ with $\mu(\bigcap U) = 0$. As $r$ is random over $M$, the pair $(r_0, r_1)$ cannot belong to $\bigcap U$, thus, there is $j \in \omega$ such that $(r_0, r_1) \notin u_j$. Then we have $(r_0, r_1) \notin u_{jk}$, i.e. $r_0|N_{jk} \neq r_1|N_{jk}$ for every $k \in \omega$. □

Remark. In view of Theorem 2 in [2], which says that the local independence of Boolean subalgebras is equivalent to the separation of corresponding cogeneric extensions, Theorem 3.3 brings another proof that the quadratic product of algebras $\mathscr{R}$, $\mathscr{R}$ is not locally independent.

If $M$ is a transitive model of ZFC, we say that the scale condition over $M$ holds if, for any function $g \in {}^{\omega}\omega$, there is a function $f \in {}^{\omega}\omega \cap M$ such that $f(n) > g(n)$ holds for any $n \in \omega$. Without loss of generality it can be assumed that $f$ is increasing.

THEOREM 3.4. *Let $M$ be a transitive model of* ZFC, *let each of real numbers $r_0$, $r_1$ be Cohen or random over $M$. If the scale condition over $M$ holds, then there exists disjoint subsets $R_0, R_1 \subseteq \omega$, $R_0 \in M(r_0)$, $R_1 \in M(r_1)$ which are not separated over $M$.*

Proof. If reals $r_0, r_1 \in {}^{\omega}2$ differ only on finitely many places, the theorem is trivial, because then $M(r_0) = M(r_1) \neq M$ holds. So, we may suppose that there exists an increasing function $g \in {}^{\omega}\omega$ such that for any $n \in \omega$ we have $r_0(g(n)) \neq r_1(g(n))$. Let $f \in {}^{\omega}\omega \cap M$ be the increasing function whose existence follows from the scale condition. Then it is

$$f(n) > g(n) \geqslant n, \qquad f(f(n)) > g(f(n)) \geqslant f(n)$$

for every $n \in \omega$. By induction through $k \in \omega$ it is easy to show that $f^{k+1}(n) > g(f^k(n)) \geqslant f^k(n)$ for every $n$, $k \in \omega$. If we set

$$N_k = \{m \in \omega;\ f^{k+1}(0) > m \geqslant f^k(0)\}$$

for $k \in \omega$, then we have $g(f^k(0)) \in N_k$, which gives $r_0|N_k \neq r_1|N_k$ for any $k \in \omega$. By Theorem 3.1 the proof is finished.

It is well-known (see e.g. [1]), that if $\mathscr{B}$ is a weakly $\omega$-distributive complete Boolean algebra in $M$ and if $N$ is a generic extension of $M$ by $\mathscr{B}$, then, in $N$, the scale condition over $M$ holds. From this fact we get the following consequence of Theorem 3.4.

THEOREM 3.5. *The minimal complete independent products $\mathscr{C} \otimes \mathscr{C}$, $\mathscr{R} \otimes \mathscr{R}$, $\mathscr{C} \otimes \mathscr{R}$ (which are also the minimal $\omega$-products) are not weakly $\omega$-distributive.*

Proof. Any minimal complete independent product is locally independent and, therefore, the corresponding extensions $M(r_0)$, $M(r_1)$ are separated over $M$. □

Remark. Theorem 3.5 strengthens the result of Theorem 2.5 in the sense that the minimal and the quadratic product of algebras $\mathscr{R}$, $\mathscr{R}$ are not only incomparable (and, therefore, non-isomorphic) as products, but that they are incomparable as algebras, because a homomorphic image of weakly

$\omega$-distributive algebra $\mathscr{R}$ cannot be non-weakly $\omega$-distributive algebra $\mathscr{R}\otimes\mathscr{R}$. The products $\mathscr{C}\otimes\mathscr{C}$, $\mathscr{C}\otimes\mathscr{R}$ are not weakly $\omega$-distributive also for that reason that they contain $\mathscr{C}$ as a subalgebra. Recently, M. Kutylowski proved that $\mathscr{R}\otimes\mathscr{R}$ also contains $\mathscr{C}$.

R. Solovay proved an (unpublished) result, called "two-Cohen theorem": *if $M$ is a transitive model class and the continuum (in the sense of $M$) $({}^{\omega}2)^M$, is countable, then any real $x$ can be represented as a sum $x = r_0 + r_1$ of two Cohen numbers $r_0$, $r_1$ over $M$.* For random numbers analogous "two-random theorem" is known. In [4], Boolean versions of these theorems are given in which the so-called collapse algebra $\mathrm{Col}\,\alpha$ is represented as a complete Boolean independent product (collapse product) of algebras $\mathscr{C}$, $\mathscr{C}$ or $\mathscr{R}$, $\mathscr{R}$.

As an application of Theorem 3.2 we bring a result concerning the two-Cohen and the two-random theorem and the collapse products.

THEOREM 3.6. *Let $M$ be a transitive model of* ZFC, *let each of real numbers $r_0$, $r_1$ be Cohen or random over $M$. If the real $x = r_0 + r_1$ codes over $M$ a well-ordering of $\omega$, then the extensions $M(r_0)$, $M(r_1)$ are not disjoint over $M$.*

Proof. By assumption, there is a bijection $f \in M$, $f\colon \omega \to \omega \times \omega$ such that $f$ translates $x \in {}^{\omega}2$ to a characteristic function $y \in {}^{\omega\times\omega}2$ of a well-ordering $\dashv$ of $\omega$. Let $n \in \omega$ be the least element in the sense of $\dashv$. Then for any $k \in \omega$ we have $n \dashv k$, i.e. $y(n, k) = 1$ and the infinite set $a = f^{-1}(\{n\} \times \omega)$ has the property described in Theorem 3.2. $\square$

Remark. As a consequence of Theorem 3.6, the collapse products of algebras $\mathscr{C}$, $\mathscr{C}$ or $\mathscr{R}$, $\mathscr{R}$ are not locally disjoint. The mixed collapse products of algebras $\mathscr{C}$, $\mathscr{R}$ do not exist at all.

## *REFERENCES*


[1] L. Bukovský, *V-model and distributivity in Boolean algebras*, Commentationes Mathematicae Universitatis Carolinae 9 (1968), p. 595–612.

[2] –, *Cogeneric extensions*, Proceedings of Wrocław Logic Colloquium 1977, North-Holland, Amsterdam 1978, p. 91–98.

[3] –, *Cogeneric extensions. II*, to appear.

[4] M. Gavalec, *Independent complete subalgebras of collapsing algebras*, Colloquium Mathematicum 45 (1981), p. 181–189.

[5] –, *Iterated products of ideals of Borel sets*, ibidem 50 (1985), p. 39–52.

[6] T. Jech, *Lectures in set theory with particular emphasis on the method of forcing*, Springer-Verlag, Berlin 1971.

[7] R. Sikorski, *Boolean algebras*, 2nd edition, Springer-Verlag, Berlin 1964.

[8] R. Solovay, *New proof of a theorem of Gaifman and Hales*, Bulletin of the American Mathematical Society 72 (1966), p. 282–284.

[9] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Annals of Mathematics 92 (1970), p. 1–56.
[10] F. Šik, *Die Anwendung der Polarität auf die direkten Produktzerlegungen einer Gruppe*, Czechoslovak Mathematical Journal 5 (1955), p. 61 75.

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