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CARATHÉODORY'S AND HELLY'S DIMENSIONS OF PRODUCTS OF CONVEXITY STRUCTURES

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Generalizing a Carathéodory's type theorem [9] and a theorem of [8] concerning the exact value of Carathéodory's dimension for the product of families of d-convex sets we establish the exact value of Carathéodory's dimension for the finite product of convexity structures. Moreover, generalizing the result of [6] we give the exact value of Helly's dimension for the extended product of convexity structures. An analogous question concerning Radon's dimension is not answered, a certain inequality is only known [3]. Also c-independence and r-independence have not been studied yet in products of convexity structures, unlike a characterization of h-independence obtained in Lemma 11.

1. Definitions and preliminaries. A multiplicative family $\mathscr{C}$ of subsets of a set X such that $\emptyset \in \mathscr{C}$ is called a *convexity structure* (cf. [3] and [5]). Sets from $\mathscr{C}$ will be called *$\mathscr{C}$-convex*. Obviously, the set X is $\mathscr{C}$-convex as the intersection of the empty family of $\mathscr{C}$-convex sets. If $\mathscr{C}$ contains a set different from X and from $\emptyset$, then $\mathscr{C}$ is said to be a *non-trivial convexity structure.*

For an arbitrary $A \subset X$ we define *$\mathscr{C}$-hull* of A (denoted by $\mathscr{C}$-$\operatorname{conv} A$) as the intersection of all $\mathscr{C}$-convex sets containing A.

If $\mathscr{C}_\lambda \subset 2^{X_\lambda}$, $\lambda \in \Lambda$, are convexity structures, then the family

$$\prod_{\lambda\in\Lambda}\mathscr{C}_\lambda = \Big\{\prod_{\lambda\in\Lambda} C_\lambda\colon\ C_\lambda \in \mathscr{C}_\lambda\Big\}$$

is a convexity structure of subsets of the set $\prod_{\lambda\in\Lambda} X_\lambda$. We call $\prod_{\lambda\in\Lambda}\mathscr{C}_\lambda$ the *product of convexity structures* $\mathscr{C}_\lambda$, $\lambda \in \Lambda$.

Let p_λ denote the projection from $\prod_{\lambda\in\Lambda} X_\lambda$ onto X_λ, $\lambda \in \Lambda$. Obviously,

$$\Big(\prod_{\lambda\in\Lambda}\mathscr{C}_\lambda\Big)\text{-}\operatorname{conv} A = \prod_{\lambda\in\Lambda}\big(\mathscr{C}_\lambda\text{-}\operatorname{conv} p_\lambda(A)\big). \tag{1}$$

By *Carathéodory's dimension* $\operatorname{cim}\mathscr{C}$ of $\mathscr{C}$ we mean the smallest integer $t \geqslant -1$ such that, for an arbitrary $Y \subset X$ and $y \in \mathscr{C}$-$\operatorname{conv} Y$, there exist

$t+1$ elements (not necessarily different) of Y such that y belongs to the $\mathscr{C}$-hull of these elements. If such an integer t does not exist, then we put $\operatorname{cim}\mathscr{C} = \infty$ (cf. [2], p. 160).

Helly's dimension $\operatorname{him}\mathscr{C}$ of $\mathscr{C}$ is the smallest integer $t \geqslant -1$ for which any finite family $\mathscr{K}$ has the non-empty intersection provided the intersection of any $t+1$ sets from $\mathscr{K}$ is non-empty. If such an integer does not exist, then we put $\operatorname{him}\mathscr{C} = \infty$.

Radon's dimension $\operatorname{rim}\mathscr{C}$ of $\mathscr{C}$ is the smallest integer $t \geqslant -1$ such that any set $Y \subset X$ containing at least $t+2$ elements can be divided into two subsets S and P satisfying

$$S \cap P = \emptyset \quad \text{and} \quad \mathscr{C}\text{-conv}\, S \cap \mathscr{C}\text{-conv} P \neq \emptyset.$$

If such an integer t does not exist, then we put $\operatorname{rim}\mathscr{C} = \infty$.

In many papers one can find characteristics larger by 1 than the above ones and called *Carathéodory's*, *Helly's*, and *Radon's numbers* (see, e.g., [5]). The advantage of our definitions is that the dimensions of the family of convex (in the usual sense) sets in an n-dimensional Euclidean space are equal to n. Obviously, the dimensions are equal to -1 if and only if $X = \emptyset$.

The following kinds of independence play an analogous role for the dimensions mentioned above as the linear independence plays for the usual dimension of Euclidean space.

A set $T \subset X$ is called: *c-independent* if

$$\mathscr{C}\text{-conv} T \neq \bigcup_{a \in T} \mathscr{C}\text{-conv}(T \setminus \{a\}) \quad \text{or} \quad T = \emptyset,$$

h-independent if

$$\bigcap_{a \in T} \mathscr{C}\text{-conv}(T \setminus \{a\}) = \emptyset \quad \text{or} \quad T = \emptyset,$$

r-independent if $T = P \cup S$ and $P \cap S = \emptyset$ imply

$$\mathscr{C}\text{-conv} P \cap \mathscr{C}\text{-conv}\, S = \emptyset.$$

We say that elements of T are *c-independent*, *h-independent* or *r-independent*, respectively. Otherwise, we call them *c-dependent*, *h-dependent* or *r-dependent*, respectively.

These three kinds of independence have been defined and investigated for d-convexity and for convexity structures in [7]. Later, on the basis of [8] (where [7] is quoted as "to appear") and of the manuscript of [7], the results have been repeated in [10]. It should be added that [10] contains also new Theorems 7-9 and 13 the meaning of which had to be supported by Theorem 6. However, Theorem 6 of [10] is not true, as can be seen by taking the structure $\mathscr{C} = \{Y \subset X\colon \operatorname{card} Y \leqslant k+1\} \cup \{X\}$

with $\operatorname{card} X > k+2$ and an arbitrary set $A \subset X$ consisting of $k+2$ elements. Moreover, the following theorems in the formulation of [10] are also false: Theorems 1.2 and 2 (for Euclidean spaces with usual convexity), Theorem 12.1 and left inequalities of Theorems 12.2, 12.3, 13 (for the product of $X_1 = \emptyset$ and X_2 being a Euclidean space with usual convexity), the right inequality of Theorem 12.2 (for the product of two 3-dimensional Euclidean spaces with usual convexities).

From generalizations in [7], p. 65, we get the following

LEMMA 1. *Carathéodory's* (*respectively, Helly's, Radon's*) *dimension of an arbitrary convexity structure* $\mathscr{C}$ *is equal to the greatest integer* t *for which there exists a set of* $t+1$ *c-independent* (*respectively, h-independent, r-independent*) *elements; if such an integer* t *does not exist, then that dimension is equal to* ∞.

Besides Carathéodory's dimension we can consider a similar number $k(\mathscr{C})$ which deals with an improvement [1] of Carathéodory's theorem.

We define $k(\mathscr{C})$ as the smallest integer $k \geqslant 0$ such that for arbitrary $x_0 \in X$, $A \subset X$, and $a \in \mathscr{C}\text{-conv}\, A$ there exist k elements (not necessarily different) $x_1, \ldots, x_k \in A$ for which $a \in \mathscr{C}\text{-conv}\{x_0, \ldots, x_k\}$. If such an integer k does not exist, then we put $k(\mathscr{C}) = \infty$.

The relation between $\operatorname{cim}\mathscr{C}$ and $k(\mathscr{C})$ depends on the condition [1]

$$(2) \qquad \bigwedge_{\substack{x_0,\ldots,x_c \\ x\in X}} \mathscr{C}\text{-conv}\{x_0, \ldots, x_c\} \subset \bigcup_{j=0}^{c} \mathscr{C}\text{-conv}\{x_0, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_c\},$$

where $c = \operatorname{cim}\mathscr{C}$ and $0 \leqslant c < \infty$.

The relation is expressed by the following obvious lemma:

LEMMA 2. *If a convexity structure* $\mathscr{C}$ *with* $0 \leqslant \operatorname{cim}\mathscr{C} < \infty$ *satisfies condition* (2), *then* $k(\mathscr{C}) = \operatorname{cim}\mathscr{C}$. *Otherwise,* $k(\mathscr{C}) = 1+\operatorname{cim}\mathscr{C}$. *Moreover,* $k(\mathscr{C}) = \infty$ *if and only if* $\operatorname{cim}\mathscr{C} = \infty$.

The family of all convex sets in a Euclidean space satisfies condition (2) (see [1]). However, for many convexity structures the condition does not hold (see, e.g., Lemma 10).

2. Carathéodory's dimension for products of convexity structures.

THEOREM 1. *If exactly* m *among* $n \geqslant 2$ *non-trivial convexity structures* $\mathscr{C}_i \subset 2^{X_i}$ $(i = 1, \ldots, n)$ *do not satisfy condition* (2), *then*

$$\operatorname{cim}\prod_{i=1}^{n} \mathscr{C}_i = m-1+\sum_{i=1}^{n} \operatorname{cim}\mathscr{C}_i.$$

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(1) The set $\{x_0, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_c\}$ in condition (2) is to be understood as $\{x, x_1, \ldots, x_c\}$ for $j = 0$ and as $\{x_0, \ldots, x_{c-1}, x\}$ for $j = c$. The same remarks relate to other parts of this paper.

It is convenient to prove Theorem 1 with the help of some lemmas. Let

$$X^* = \prod_{i=1}^{n} X_i, \quad \mathscr{C}^* = \prod_{i=1}^{n} \mathscr{C}_i, \quad c_i = \operatorname{cim}\mathscr{C}_i \ (i = 1, \ldots, n), \quad c^* = \operatorname{cim}\mathscr{C}^*.$$

If at least one of the dimensions c_i $(i = 1, \ldots, n)$ equals ∞, then Theorem 1 is obvious. Let $0 \leqslant c_i < \infty$ for $i = 1, \ldots, n$. In Lemmas 3-9 we consider the case $n = 2$.

LEMMA 3. $c^* \leqslant c_1 + c_2 + 1$.

Proof. Let $S \subset X^*$ and $s \in \mathscr{C}^*$-conv$S$. By (1) and the definition of Carathéodory's dimension, there exist elements $x_0, \ldots, x_{c_1}, y_0, \ldots, y_{c_2} \in S$ such that

$$p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(x_0), \ldots, p_1(x_{c_1})\}, \quad p_2(s) \in \mathscr{C}_2\text{-conv}\{p_2(y_0), \ldots, p_2(y_{c_2})\}.$$

From (1) we infer that $s \in \mathscr{C}^*\text{-conv}\{x_0, \ldots, x_{c_1}, y_0, \ldots, y_{c_2}\}$. Hence any element $s \in \mathscr{C}^*$-conv$S$ belongs to the $\mathscr{C}^*$-hull of not more than $c_1 + c_2 + 2$ elements of S. Therefore, the inequality $c^* \leqslant c_1 + c_2 + 1$ is satisfied.

LEMMA 4. *If* (2) *holds for at least one of the structures* $\mathscr{C}_1$ *and* $\mathscr{C}_2$, *then* $c^* \leqslant c_1 + c_2$.

Proof. We use the notation of Lemma 3. It is sufficient to show that s belongs to the $\mathscr{C}^*$-hull of not more than $c_1 + c_2 + 1$ elements of S. Let $\mathscr{C}_1$ satisfy (2). Since $p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(x_0), \ldots, p_1(x_{c_1})\}$, we have

$$p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(x_0), \ldots, p_1(x_{j-1}), p_1(y_0), p_1(x_{j+1}), \ldots, p_1(x_{c_1})\}$$

for some $j \in \{0, \ldots, c_1\}$. Hence

$$s \in \mathscr{C}^*\text{-conv}\{x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{c_1}, y_0, \ldots, y_{c_2}\}.$$

LEMMA 5. *If both structures* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *are non-trivial and satisfy condition* (2), *then* $c^* \leqslant c_1 + c_2 - 1$.

Proof. We use the notation of Lemma 3. Since $\mathscr{C}_1$ is non-trivial and fulfils (2), $c_1 \geqslant 1$. Similarly, $c_2 \geqslant 1$. It is sufficient to show that s belongs to the $\mathscr{C}^*$-hull of not more than $c_1 + c_2$ elements of S.

Case 1. Not all elements $x_0, \ldots, x_{c_1}$ are different or not all elements $y_0, \ldots, y_{c_2}$ are different.

Let, for instance, not all elements $y_0, \ldots, y_{c_2}$ be different. We can assume $y_0 = y_1$. Since (2) is satisfied for $\mathscr{C}_1$, for some $i \in \{0, \ldots, c_1\}$ we have

$$p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(x_0), \ldots, p_1(x_{i-1}), p_1(y_1), p_1(x_{i+1}), \ldots, p_1(x_{c_1})\}.$$

Therefore,

$$s \in \mathscr{C}^*\text{-conv}\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{c_1}, y_1, \ldots, y_{c_2}\},$$

i.e., s belongs to the $\mathscr{C}^*$-hull of not more than $c_1 + c_2$ elements of S.

Case 2. All elements $x_0, \ldots, x_{c_1}$ are different and all elements $y_0, \ldots, y_{c_2}$ are different, and $c_1 \geqslant 2$ (or $c_2 \geqslant 2$).

Since $\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy (2) and $c_1 \geqslant 2$, there exist numbers j', j'', and j $(j \neq j', j \neq j'')$ in the set $\{0, \ldots, c_1\}$ and a number i in the set $\{0, \ldots, c_2\}$ such that

(a) $p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(x_0), \ldots, p_1(x_{j'-1}), p_1(y_0), p_1(x_{j'+1}), \ldots, p_1(x_{c_1})\}$,

(b) $p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(x_0), \ldots, p_1(x_{j''-1}), p_1(y_1), p_1(x_{j''+1}), \ldots, p_1(x_{c_1})\}$,

(c) $p_2(s) \in \mathscr{C}_2\text{-conv}\{p_2(y_0), \ldots, p_2(y_{i-1}), p_2(x_j), p_2(y_{i+1}), \ldots, p_2(y_{c_2})\}$.

If $i \neq 0$, then from the conditions (1), $x_{j'} \neq x_j$, (a), and (c) we get

$$s \in \mathscr{C}^*\text{-conv}\{x_0, \ldots, x_{j'-1}, x_{j'+1}, \ldots, x_{c_1}, y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{c_2}\}.$$

If $i = 0$, then from the conditions (1), $x_{j''} \neq x_j$, (b), and (c) it follows that

$$s \in \mathscr{C}^*\text{-conv}\{x_0, \ldots, x_{j''-1}, x_{j''+1}, \ldots, x_{c_1}, y_1, \ldots, y_{c_2}\}.$$

Case 3. $c_1 = c_2 = 1$.

By Lemma 4 we have $c^* \leqslant 2$. Hence s belongs to the $\mathscr{C}^*$-hull of some three elements u, w, v of S. From (1) it follows that

$$p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(u), p_1(w), p_1(v)\}.$$

Since $c_1 = 1$, $p_1(s)$ belongs to at least one of the following sets:

$$\mathscr{C}_1\text{-conv}\{p_1(u), p_1(w)\}, \quad \mathscr{C}_1\text{-conv}\{p_1(u), p_1(v)\}, \quad \mathscr{C}_1\text{-conv}\{p_1(w), p_1(v)\}.$$

Actually, $p_1(s)$ belongs to at least two of these sets because $\mathscr{C}_1$ satisfies (2). Similarly, it can be shown that $p_2(s)$ belongs to at least two of the following three sets:

$$\mathscr{C}_2\text{-conv}\{p_2(u), p_2(w)\}, \quad \mathscr{C}_2\text{-conv}\{p_2(u), p_2(v)\},$$
$$\mathscr{C}_2\text{-conv}\{p_2(w), p_2(v)\}.$$

Therefore, we can choose two among the elements u, w, v (suppose u and w) for which

$$p_1(s) \in \mathscr{C}_1\text{-conv}\{p_1(u), p_1(w)\} \quad \text{and} \quad p_2(s) \in \mathscr{C}_2\text{-conv}\{p_2(u), p_2(w)\}.$$

From (1) it follows that $s \in \mathscr{C}^*\text{-conv}\{u, w\}$.

LEMMA 6. $c^* \geqslant c_1 + c_2 - 1$.

Proof. From Lemma 1 it follows that for any convexity structure $\mathscr{C} \subset 2^X$ with $c = \text{cim}\mathscr{C}$ there exist different elements $d_0, \ldots, d_c \in X$ and an element $d' \in X$ for which we have

$$d' \in \mathscr{C}\text{-conv}\{d_0, \ldots, d_c\}, \tag{3}$$

$$d' \notin \mathscr{C}\text{-conv}\{d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_c\}, \quad i = 0, \ldots, c. \tag{4}$$

Assume that $a_0, \ldots, a_{c_1}, a'$ and $b_0, \ldots, b_{c_2}, b'$ are such elements in the cases of $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively. The set

$$T = \{(a_0, b_1), \ldots, (a_0, b_{c_2}), (a_1, b_0), \ldots, (a_{c_1}, b_0)\}$$

consists of c_1+c_2 different elements of the set $X^* = X_1 \times X_2$. From (3) and (1) we get $(a', b') \in \mathscr{C}^*\text{-conv}T$. Let $P \subsetneq T$. We shall show that $(a', b') \notin \mathscr{C}^*\text{-conv}P$. Let $(a_0, b_i) \notin P$, where $i \in \{1, \ldots, c_2\}$. From (4) we get

$$b' \notin \mathscr{C}_2\text{-conv}\{b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{c_1}\} \supset \mathscr{C}_2\text{-conv}p_2(P).$$

By (1) we get $(a', b') \notin \mathscr{C}^*\text{-conv}P$. If $(a_j, b_0) \notin P$, then the consideration is analogous. Hence the set T is c-independent. Since T consists of c_1+c_2 elements, by Lemma 1 we have $c^* \geqslant c_1+c_2-1$.

LEMMA 7. *If both structures $\mathscr{C}_1$ and $\mathscr{C}_2$ do not satisfy condition* (2), *then* $c^* \geqslant c_1+c_2+1$.

Proof. If for the structure $\mathscr{C}$ with $c = \text{cim}\mathscr{C}$ condition (2) is not satisfied, then there exist different elements $d_0, \ldots, d_c, d \in X$ and an element $d' \in X$ for which condition (3) and the following condition hold:

$$d' \notin \mathscr{C}\text{-conv}\{d_0, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_c\}, \quad i = 0, \ldots, c. \tag{5}$$

Assume that $a_0, \ldots, a_{c_1}, a, a'$ and $b_0, \ldots, b_{c_2}, b, b'$ are such elements in the cases of $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively. Therefore the set

$$T = \{(a, b_0), \ldots, (a, b_{c_2}), (a_0, b), \ldots, (a_{c_1}, b)\}$$

consists of c_1+c_2+2 different elements. Similarly as in Lemma 6, we can show that the element (a', b') belongs to the set $\mathscr{C}^*\text{-conv}T$ and does not belong to the $\mathscr{C}^*$-hull of any c_1+c_2+1 elements of T. Hence T is c-independent. Since T consists of c_1+c_2+2 different elements, by Lemma 1 we get $c^* \geqslant c_1+c_2+1$.

LEMMA 8. *If at least one of the structures $\mathscr{C}_1$ and $\mathscr{C}_2$ does not satisfy condition* (2), *then* $c^* \geqslant c_1+c_2$.

Proof. Assume that $\mathscr{C}_1$ does not satisfy condition (2). For the same reasons as in Lemma 7, there exist different elements $a_0, \ldots, a_{c_1}, a \in X_1$ and an element $a' \in X_1$ for which (3) and (5) hold. Analogously as in Lemma 6, there exist different elements $b_0, \ldots, b_{c_2} \in X_2$ and an element $b' \in X_2$ for which (3) and (4) are satisfied. Hence the set

$$T = \{(a, b_1), \ldots, (a, b_{c_2}), (a_0, b_0), \ldots, (a_{c_1}, b_0)\}$$

consists of c_1+c_2+1 different elements. Similarly as in the proof of Lemma 7, it is easy to show that (a', b') belongs to the set $\mathscr{C}^*\text{-conv}T$ and does not belong to the $\mathscr{C}^*$-hull of any c_1+c_2 elements of T. Therefore, T is c-independent and, consequently, the inequality $c^* \geqslant c_1+c_2$ is satisfied.

As an immediate consequence of Lemmas 3-8 we get

LEMMA 9. *For non-trivial structures* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *we have*

$$c^* = \begin{cases} c_1+c_2-1 & \textit{if (2) holds for both structures,} \\ c_1+c_2 & \textit{if (2) holds exactly for one structure,} \\ c_1+c_2+1 & \textit{if (2) does not hold for either structure.} \end{cases}$$

LEMMA 10. *Condition* (2) *does not hold for the product of any* $n \geqslant 2$ *non-trivial convexity structures.*

Proof. Obviously, it is sufficient to prove Lemma 10 for $n = 2$. To show that (2) is not satisfied for the structure $\mathscr{C}^* = \mathscr{C}_1 \times \mathscr{C}_2$ it suffices to find elements $d_0, \ldots, d_c, d, d' \in X_1 \times X_2$ for which (3) and (5) hold. We consider three cases as in Lemma 9. In the first case we put $\{d_0, \ldots, d_c\} = T$, $d = (a_0, b_0)$, $d' = (a', b')$ as in Lemma 6 (notice that $c_1 \geqslant 1$, $c_2 \geqslant 1$). In the second case let $\{d_0, \ldots, d_c\} = T, d = (a, b_0), d' = (a', b')$ as in Lemma 8 (notice that $c_2 \geqslant 1$). In the last case let $\{d_0, \ldots, d_c\} = T$, $d = (a, b)$, $d' = (a', b')$ as in Lemma 7. It is easy to see that (3) and (5) hold in every case. Therefore, condition (2) does not hold for the product $\mathscr{C}^*$.

Proof of Theorem 1 is carried out recurrently. For $n = 2$ the theorem holds by Lemma 9. Suppose that the theorem is true for the product of $r \geqslant 2$ convexity structures. Consider the product of $r+1$ structures $\mathscr{C}_i$ $(i = 1, \ldots, r+1)$. Assume that exactly m among these structures do not satisfy condition (2).

Case 1. The structure $\mathscr{C}_{r+1}$ satisfies condition (2).

Then there are exactly m among the structures $\mathscr{C}_i$ $(i = 1, \ldots, r)$ for which (2) does not hold. Since the theorem is true for $n = 2$ (see Lemma 9) and by Lemma 10, we have

$$\begin{aligned} \operatorname{cim} \prod_{i=1}^{r+1} \mathscr{C}_i &= \operatorname{cim}\Big[\Big(\prod_{i=1}^{r} \mathscr{C}_i\Big) \times \mathscr{C}_{r+1}\Big] = \operatorname{cim} \prod_{i=1}^{r} \mathscr{C}_i + c_{r+1} \\ &= \Big(m-1+\sum_{i=1}^{r} c_i\Big) + c_{r+1} = m-1+\sum_{i=1}^{r+1} c_i. \end{aligned}$$

Case 2. The structure $\mathscr{C}_{r+1}$ does not satisfy condition (2).

Then there are exactly $m-1$ among the structures $\mathscr{C}_i$ $(i = 1, \ldots, r)$ for which (2) does not hold. Consequently,

$$\begin{aligned} \operatorname{cim} \prod_{i=1}^{r+1} \mathscr{C}_i &= \operatorname{cim}\Big[\Big(\prod_{i=1}^{r} \mathscr{C}_i\Big) \times \mathscr{C}_{r+1}\Big] = \operatorname{cim} \prod_{i=1}^{r} \mathscr{C}_i + c_{r+1} + 1 \\ &= \Big(m-2+\sum_{i=1}^{r} c_i\Big) + c_{r+1} + 1 = m-1+\sum_{i=1}^{r+1} c_i. \end{aligned}$$

Thus the proof of Theorem 1 is complete.

THEOREM 2. *Let $X_i \neq \emptyset$ and let $\mathscr{C}_i \subset 2^{X_i}$ be a convexity structure, $i = 1, \ldots, n$. Then for the product $\mathscr{C}^*$ of the above structures we have*

$$k(\mathscr{C}^*) = \sum_{i=1}^{n} k(\mathscr{C}_i). \tag{6}$$

Proof. For $n = 1$ the theorem is obvious.

Let $n = 2$. For $i = 1, 2$ let $\delta_i = 0$ if (2) holds for $\mathscr{C}_i$ and let $\delta_i = 1$ otherwise. If $\mathscr{C}_1$ and $\mathscr{C}_2$ are non-trivial structures, then by Lemmas 2, 9, and 10 we get

$$k(\mathscr{C}_1)+k(\mathscr{C}_2) = \delta_1+\operatorname{cim}\mathscr{C}_1+\delta_2+\operatorname{cim}\mathscr{C}_2 = \operatorname{cim}\mathscr{C}^*+1 = k(\mathscr{C}^*).$$

Obviously, if at least one of the structures is not non-trivial, then also $k(\mathscr{C}_1)+k(\mathscr{C}_2) = k(\mathscr{C}^*)$. Therefore, the theorem holds for $n = 2$.

Now, recurrently, we get equality (6).

3. Helly's dimension of the product. Let

$$X^* = \prod_{\lambda \in \Lambda} X_\lambda \quad \text{and} \quad \mathscr{C}^* = \prod_{\lambda \in \Lambda} \mathscr{C}_\lambda.$$

LEMMA 11. *Elements $a_\omega \in X^*$, $\omega \in \Omega$, are h-independent if and only if there exists $\lambda \in \Lambda$ for which the elements $p_\lambda(a_\omega) \in X_\lambda$, $\omega \in \Omega$, are h-independent.*

Proof. From (1) we get

$$\bigcap_{\omega' \in \Omega} \mathscr{C}^*\text{-conv}\{a_\omega\colon \omega \in \Omega,\ \omega \neq \omega'\} = \bigcap_{\omega' \in \Omega} \prod_{\lambda \in \Lambda} \mathscr{C}_\lambda\text{-conv}\{p_\lambda(a_\omega)\colon \omega \in \Omega,\ \omega \neq \omega'\}$$
$$= \prod_{\lambda \in \Lambda} \bigcap_{\omega' \in \Omega} \mathscr{C}_\lambda\text{-conv}\{p_\lambda(a_\omega)\colon \omega \in \Omega,\ \omega \neq \omega'\}.$$

Thus, Lemma 11 follows from this equality, from the definition of h-independence, and from the equivalence

$$\prod_{\lambda \in \Lambda} A_\lambda = \emptyset \Leftrightarrow (A_\lambda = \emptyset \text{ for some } \lambda \in \Lambda).$$

Lemma 11 implies immediately

LEMMA 12. *Elements $a_\omega \in X^*$, $\omega \in \Omega$, are h-dependent if and only if the elements $p_\lambda(a_\omega) \in X_\lambda$, $\omega \in \Omega$, are h-dependent for any $\lambda \in \Lambda$.*

THEOREM 3. *For the product $\mathscr{C}^*$ of convexity structures $\mathscr{C}_\lambda \subset 2^{X_\lambda}$ such that $X_\lambda \neq \emptyset$ for $\lambda \in \Lambda$, we have*

$$\operatorname{him}\mathscr{C}^* = \sup_{\lambda \in \Lambda} \operatorname{him}\mathscr{C}_\lambda.$$

Proof. Let $h^* = \operatorname{him}\mathscr{C}^*$ and $h_\lambda = \operatorname{him}\mathscr{C}_\lambda$, $\lambda \in \Lambda$.

If $h^* = \infty$, then by Lemma 1 for any natural number n there exists a set $T \subset X^*$ of at least $n+1$ h-independent elements. From Lemmas 1 and 11 we get $\sup_{\lambda \in \Lambda} h_\lambda = \infty$.

Let $h^* < \infty$. By Lemma 1 there exists a set of h^*+1 h-independent elements $a_0, \ldots, a_{h^*} \in X^*$. For some $\lambda' \in \Lambda$ the elements $p_{\lambda'}(a_0), \ldots, p_{\lambda'}(a_{h^*})$ of $X_{\lambda'}$ are h-independent, which is guaranteed by Lemma 11. From Lemma 1 we get

$$h^* \leqslant h_{\lambda'} \leqslant \sup_{\lambda \in \Lambda} h_\lambda.$$

On the other hand, from Lemma 1 it follows that any h^*+2 elements of X^* are h-dependent. By Lemma 12, any h^*+2 elements of X_λ are h-dependent for any $\lambda \in \Lambda$. Hence $h_\lambda \leqslant h^*$ for all $\lambda \in \Lambda$, i.e. $\sup_{\lambda \in \Lambda} h_\lambda \leqslant h^*$. Therefore

$$h^* = \sup_{\lambda \in \Lambda} h_\lambda.$$

4. The case of d-convexity. Let X be a metric space with metric d.

A set $D \subset X$ is called *d-convex* if for any $a \in D$, $b \in X$, $c \in D$ such that $d(a, b)+d(b, c) = d(a, c)$ it follows that $b \in D$ (see [4]).

Obviously, the family $\mathscr{D}$ of d-convex sets of the space X is a convexity structure.

It is known that

$$d_\Sigma\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) = \sum_{i=1}^{n} d_i(a_i, b_i)$$

is a metric in the product $X^* = \prod_{i=1}^{n} X_i$ of metric spaces X_i with metrics d_i $(i = 1, \ldots, n)$. Moreover, if $\mathscr{D}_i$ is the family of d_i-convex sets $(i = 1, \ldots, n)$, then

$$\mathscr{D}^* = \prod_{i=1}^{n} \mathscr{D}_i$$

is the family of d_Σ-convex sets [6]. Consequently, (1), Theorem 1 with the lemmas, Theorem 2, and also finite variants of Lemmas 11 and 12 and of Theorem 3 hold for the family $\mathscr{D}^*$ of d_Σ-convex sets.

In considerations of a known theorem of Steinitz (see, e.g., [2], p. 116) we define Steinitz's number and we give its value for $\mathscr{D}^*$.

By *Steinitz's number* $s(\mathscr{D})$ of the family $\mathscr{D}$ of d-convex sets of a metric space X we mean the smallest integer t such that, for any $Y \subset X$ and $y \in \operatorname{int} d\text{-conv}\, Y$, there exist t points $y_1, \ldots, y_t \in Y$ such that $y \in \operatorname{int} d\text{-conv}\{y_1, \ldots, y_t\}$. If such an integer t does not exist, then we put $s(\mathscr{D}) = \infty$.

For the family $\mathscr{D}$ with $s = s(\mathscr{D}) < \infty$ we consider the condition

(7)

$$\bigwedge_{\substack{x_1, \ldots, x_s \\ x \in X}} \operatorname{int} d\text{-conv}\{x_1, \ldots, x_s\} \subset \bigcup_{j=1}^{s} \operatorname{int} d\text{-conv}\{x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_s\}.$$

THEOREM 4. *If exactly q among the families $\mathscr{D}_i$ of d_i-convex sets $(i = 1, \ldots, n)$ satisfy condition* (7), *then*

$$s(\mathscr{D}^*) = -q + \sum_{i=1}^{n} s(\mathscr{D}_i).$$

Proof. In the proof of Theorem 1 we use only (1) and the following isotonic property of the operation $\mathscr{C}$-conv:

$$A \subset B \Rightarrow \mathscr{C}\text{-conv}A \subset \mathscr{C}\text{-conv}B.$$

Lemma 1 used in the proofs of Lemmas 6-8 can be also generalized for isotonic operations [7]. Obviously, the operation int$d$-conv is also isotonic. Moreover, for the operation int$d$-conv an analogue of (1) is also true:

$$\operatorname{int} d_\Sigma\text{-conv}A = \operatorname{int} \prod_{i=1}^{n} d_i\text{-conv}p_i(A) = \prod_{i=1}^{n} \operatorname{int} d_i\text{-conv}p_i(A).$$

Consequently, for the number $z(\mathscr{D}) = s(\mathscr{D})-1$ we can repeat the proof of Theorem 1 without essential changes. This follows from the comparison of the definitions of Carathéodory's dimension and Steinitz's number. We obtain

$$z(\mathscr{D}^*) = m-1+\sum_{i=1}^{n} z(\mathscr{D}_i),$$

where m denotes the number of families $\mathscr{D}_i$ for which (7) does not hold. Thus

$$s(\mathscr{D}^*) = 1+z(\mathscr{D}^*) = 1+m-1+\sum_{i=1}^{n} z(\mathscr{D}_i) = m+\sum_{i=1}^{n}[s(\mathscr{D}_i)-1]$$
$$= m-n+\sum_{i=1}^{n} s(\mathscr{D}_i) = -q+\sum_{i=1}^{n} s(\mathscr{D}_i).$$

Example. From Theorems 1-4 it follows that for the family $\mathscr{D}_\Sigma$ of d_Σ-convex sets of the space R^n, where

$$d_\Sigma\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) = \sum_{i=1}^{n} |b_i - a_i|,$$

we have

$$\operatorname{cim}\mathscr{D}_\Sigma = n-1 \ (n \geqslant 2), \quad \operatorname{him}\mathscr{D}_\Sigma = 1, \quad k(\mathscr{D}_\Sigma) = n, \quad s(\mathscr{D}_\Sigma) = 2n.$$

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