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MINIMAL REALCOMPACT SPACES

BY

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1. Introduction. If π is a topological property, a topological space $(X, \boldsymbol{T})$ is called *minimal* π if $\boldsymbol{T}$ has property π and any π-topology weaker than $\boldsymbol{T}$ coincides with $\boldsymbol{T}$. $(X, \boldsymbol{T})$ is *π-closed* if $\boldsymbol{T}$ has property π and $(X, \boldsymbol{T})$ is a closed subspace of every π-space in which it is imbedded. Extensive studies of such spaces for the cases π = Hausdorff, first countable Hausdorff, regular, first countable regular, Urysohn, first countable Urysohn, completely regular, first countable completely regular, completely Hausdorff, normal, first countable normal, completely normal, first countable completely normal, perfectly normal, first countable perfectly normal, paracompact, first countable paracompact, zero-dimensional or metric are to be found in [1]-[4], [7]-[9] and [11]. In this article we study minimal π and π-closed spaces for π = realcompact or first countable realcompact. Our investigation will show that minimal (first countable) realcompactness, (first countable) realcompact-closedness and (first countable) compactness are equivalent conditions.

All separation properties assume Hausdorff property. The definition and characterisations of realcompactness, as used here, are from Engelking [5]. The term "space" always means a topological space.

2. Notation and definitions. Let $(X, \boldsymbol{T})$ be a (first countable) realcompact space.

$F(X)$ denotes the space of all continuous functions from $(X, \boldsymbol{T})$ into $[0, 1]$ with the usual topology.

For $x \in X$, $\boldsymbol{B}(x)$ stands for the neighbourhood filter of x in $(X, \boldsymbol{T})$.

An open filter base $\boldsymbol{C}$ on a space X is called *completely regular* if for each $C \in \boldsymbol{C}$ there exists $D \in \boldsymbol{C}$ and $f \in F(X)$ such that $D \subset C$ and $f(D) = 0$ and $f(X - C) = 1$. For such a filter base

$$\bigcap \boldsymbol{C} = \bigcap \{C : C \in \boldsymbol{C}\} = \bigcap \{\bar{C} : C \in \boldsymbol{C}],$$

where $\bar{C}$ denotes the closure of C in X. A completely regular filter base $\boldsymbol{C}$ is *fixed* (*free*) if $\bigcap \boldsymbol{C} \neq \emptyset$ $(\bigcap \boldsymbol{C} = \emptyset)$.

3. Results. Let us first note the well-known fact that in the presence of realcompactness a space is compact if and only if it is pseudocompact. Hence whenever we shall try to show a realcompact space to be compact we need to show pseudocompactness only. In [10], p. 438, it has been established that a topological space Y is pseudocompact if and only if every countable completely regular filter base on Y is fixed.

THEOREM 1. *Let* $(X, \boldsymbol{T})$ *be a topological space. The following are equivalent*:

(i) X *is minimal realcompact.*

(ii) X *is realcompact-closed.*

(iii) X *is compact Hausdorff.*

THEOREM 2. *Let* $(X, \boldsymbol{T})$ *be a topological space. The following are equivalent*:

(i) X *is minimal first countable realcompact.*

(ii) X *is first countable realcompact-closed.*

(iii) X *is first countable compact Hausdorff.*

We give the proof of Theorem 2. To prove Theorem 1 one paraphrases the same proof ignoring all allusions to first countability.

Proof of Theorem 2. (i) $\Rightarrow$ (ii). We assume that $(X, \boldsymbol{T})$ is minimal first countable realcompact. To show that $(X, \boldsymbol{T})$ is first countable realcompact-closed, let Y be a first countable realcompact space containing X as a subset. We want to show that X is closed in Y. Let q belong to the closure of X in Y. Let $\boldsymbol{N}$ denote a countable fundamental system of open neighbourhoods of q in Y. Then $\boldsymbol{C} = \boldsymbol{N} \mid X$ is a countable completely regular filter base on X. Suppose $q \notin X$, i. e., X is not closed in Y, then $\bigcap \boldsymbol{C} = \emptyset$, i. e., $\boldsymbol{C}$ is free. Let $\boldsymbol{C} = \{C_n : n \geqslant 1\}$. Fix $x_0 \in X$. Consider the topology $\boldsymbol{T}'$ on X generated by the following neighbourhoods:

$$\boldsymbol{B}'(x) = \boldsymbol{B}(x) \quad \text{if } x \neq x_0,$$

$$\boldsymbol{B}'(x_0) = \{V \cup C_n : n \geqslant i,\ V \in \boldsymbol{B}(x_0)\}.$$

By Lemma 2.8 of [11], $(X, \boldsymbol{T}')$ is a first countable completely regular Hausdorff space, and $\boldsymbol{T}'$ is strictly weaker than $\boldsymbol{T}$.

$(X, \boldsymbol{T}')$ *is realcompact.* First, let

$$D_0'(X) = \{f^{-1}\{0\} : f \in F'(X)\}$$

be the set of all zero-sets of $(X, \boldsymbol{T}')$, where $F'(X)$ is the space of all continuous functions $(X, \boldsymbol{T}') \to [0, 1]$. Let us consider a z-ultrafilter $\boldsymbol{G}'$ on $(X, \boldsymbol{T}')$ which has the countable intersection property (i. e., a maximal subfamily $\boldsymbol{G}'$ of $D_0(X)$ with respect to the countable intersection property).

We wish to show that G' is fixed (i. e., $\bigcap G' \neq \emptyset$). If $x_0 \in \cap G'$, there is nothing to prove. Let us suppose then that there exists a set $G' \in G'$ with $x_0 \notin G'$. Since $T' \subset T$, it follows from Zorn's lemma that there is a Z-ultrafilter G on (X, T) (the terminology is after Gillman and Jerison [6]) such that $G' \subset G$. If G has the countable intersection property, then, by the realcompactness of (X, T), G and, a fortiori, G' are fixed. So the proof will be complete if we show that G has the countable intersection property.

Let $\{Z_i\colon i \geqslant 1\} \subset G$. We may assume that $Z_i = Z(f_i)$, where $f_i \in F(X)$. Also we may suppose that $G' = Z(f) \in D_0'(X)$ such that $f \in F'(X)$ and f equals 1 on some T'-neighbourhood of x_0 (simple computation will yield this). If we set $h_i = f \vee f_i$, then $h_i \in F'(X)$ and $h_i^{-1}\{0\} = Z(h_i) = Z(f) \cap Z_i = G' \cap Z_i$. So each $G' \cap Z_i \in D_0'(X)$, i. e., it is a zero-set of (X, T'). Now, for each $i \geqslant 1$, $G' \cap Z_i$ has non-empty intersection with every member of the Z-ultrafilter G' and, hence, $G' \cap Z_i \in G'$. Since G' has the countable intersection property,

$$\bigcap_{i=1}^{\infty} (G' \cap Z_i) = G' \cap \Big(\bigcap_{i=1}^{\infty} Z_i\Big) \in G' \subset G$$

and, inasmuch as G is a Z-ultrafilter, $\bigcap_{i=1}^{\infty} Z_i \in G$. Thus the realcompactness of (X, T') has been established.

Now, we see that (X, T') is a strictly weaker first countable realcompact space than (X, T). This contradicts the hypothesis about (X, T). We are then forced to conclude that $q \in X$, i. e., X is closed in Y.

(ii) $\Rightarrow$ (iii). We assume that (X, T) is first countable realcompact-closed. We have to show that (X, T) is compact. Since we need only to show pseudocompactness, we start with a countable completely regular filter base $C = \{C_m : m \geqslant 1\}$ (pseudocompactness is equivalent to the property that every countable completely regular filter base is fixed). Suppose C is free, i. e., $\bigcap C = \emptyset$. Take $p \notin X$, let $Y = X \cup \{p\}$, and call a set $V \subset Y$ open if and only if (a) $V \cap X$ is open in X, and (b) if $p \in V$, then V contains a set from C.

Claim. *Y is a first countable realcompact space in which X is imbedded as a proper dense subset.*

In fact, ([11], p. 118) it has been shown that Y is a first countable completely regular space in which X is dense. Note that $\{p\}$ is a compact subset of Y and X is a realcompact subspace of Y. Conclusion that Y is realcompact is now obtained from the following theorem of [6], p. 121:

In any completely regular space, the union of a compact set with a realcompact set is realcompact.

The claim is thus justified. But this contradicts the first countable realcompact-closedness of (X, T). Hence C must be fixed.

(iii) $\Rightarrow$ (i). By hypothesis, $(X, \boldsymbol{T})$ is a compact Hausdorff first countable space and hence first countable realcompact. The well-known fact that a compact Hausdorff space cannot admit any strictly weaker Hausdorff topology implies that $(X, \boldsymbol{T})$ ought to be minimal first countable realcompact, q. e. d.

Finally, we observe some easy consequences of Theorems 1 and 2.

COROLLARY 1. *All closed subspaces of a minimal (first countable) realcompact space are minimal (first countable) realcompact.*

Since every compact Hausdorff space is of the second category, the following is immediate:

COROLLARY 2. *Any minimal (first countable) realcompact space is of the second category.*

4. Remarks. It is interesting to note that if the first countability is replaced by the second countability our Theorem 2 still holds good and the same proof works without significant modifications.

At the end we wish to mention that a realcompact space may not be stronger than a minimal realcompact topology. The following is an example:

X = Rational numbers, $\boldsymbol{T}$ = Usual topology on X.

$(X, \boldsymbol{T})$ is a realcompact space. Since X is countable and no point of X is isolated in the topology $\boldsymbol{T}$, $\boldsymbol{T}$ cannot contain any compact Hausdorff topology. So there exists no minimal realcompact topology weaker than $\boldsymbol{T}$ on the set of rationals.

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