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HOMOTOPY GROUPS OF ARC COMPLEMENTS IN S^n OR Q

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1. Introduction and terminology. The main concern of this note is the homotopy groups of open n-manifolds obtained by removing wild arcs from the n-sphere S^n. In order to fully motivate our results, we discuss these matters and related historical details as follows.

The classic work of Fox and Artin [8] initiates the study of wildly embedded arcs in S^3 (an *arc* is a space which is homeomorphic to the closed interval $[0, 1]$). They exhibit an arc A in S^3 (see Example (1.1) of [8]) which is demonstrated to be wild by showing that $\pi_1(S^3-A)$ is non-trivial, i.e., $\pi_1(S^3-A)$ *detects the wildness* of A. Subsequently, there are numerous examples of wild arcs in S^n, $n > 2$, whose wildness is detected by (the non-vanishing of) the fundamental groups of their complements (see, e.g., [1], [2], [6]). Similar results are also known when S^n is replaced by the Hilbert cube Q (see [4], [6], [15]).

Throughout this section let α be an arc in S^n, $n > 2$. It is a general fact that the reduced integral homology groups of $S^n-\alpha$ vanish (use the Alexander Duality [14]), i.e., $S^n-\alpha$ is an *acyclic* n-manifold. It is reasonable to ask whether $S^n-\alpha$ is also *aspherical for all* $n > 2$ (i.e., $\pi_i(S^n-\alpha)$ vanishes for all $i > 1$) since this is at least the case for $S^3-\alpha$ (use the Sphere Theorem). For any $n > 3$, there is no known example of an arc α in S^n such that $\pi_i(S^n-\alpha)$ is non-trivial for some or, more significantly, for all $i > 1$; the same applies to arcs in Q. If such an arc α exists in S^n or Q, it must *geometrically link* suitable (singular) spheres (in S^n or Q).

The purpose of this note is to present a method of constructing (wild) arcs in S^n with $n > 4$ whose complements have infinitely many non-trivial homotopy groups, and thus fail to be aspherical in a drastic manner. As an application of this method, we produce, for each $n > 4$, an infinite family of arcs α's in S^n such that *every* homotopy group of $S^n-\alpha$ is non-vanishing for each arc α. Furthermore, the algebraic structure of these groups is explicitly determined. An added advantage of our method is that we have analogous results when S^n is replaced by the Hilbert cube Q. Precise statements can be found in Theorems (3.4) and (4.3) (see also (4.1.1)). In this way, we construct for each $n > 4$ an infinite family of smooth, acyclic, and open n-manifolds (as arc complements in S^n) whose every homotopy group is non-trivial. Since the complement of an arc in Q is a Q-manifold, many interesting examples of Q-manifolds also result. We do not pursue applications of these results to the theory of generalized n-manifolds, cell-like decompositions of S^n or Q, topological embeddings of spheres in S^n or Q, etc. (see [6] for related discussions).

This note is a continuation of our earlier work [6] with R. J. Daverman; we wish to thank him for his help and encouragement.

2. Homology spheres and arcs in S^n or Q. We refer to [3] and [5] for matters concerning homology spheres and crumpled cubes, respectively. The following is intended as a brief review of some results proved in [6].

(2.1) PROPOSITION. *For each homology n-sphere H^n with $\pi_1 H^n \not\approx \{1\}$ (of course, $n > 2$) and each integer $k > 1$, there exist a crumpled m-cube C ($m = n+k$) contained in S^m and an arc α tamely embedded in the boundary ∂C of C such that:*

(a) *the closure $S^m - C$ is an m-cell B;*

(b) *∂C is locally flat modulo α;*

(c) $\pi_1(S^m - \alpha) \approx \pi_1 H^n$;

(d) *$S^m - \alpha$, $C - \alpha$, and $H^n - \{x\}$ are homotopy equivalent.*

This proposition appears in [6], however, (d) is not explicitly stated there; it is clear from the context (see [6]). The following is a consequence of the technique of "infinite inflations" given in [4].

(2.2) PROPOSITION. *For each crumpled n-cube C such that the identity sewing (the space obtained by identifying C with itself by the identity along ∂C) is homeomorphic to S^n, there exists an embedding $e: C \to Q$ into the Hilbert cube Q such that $C - A$ has the homotopy type of $Q - \hat{A}$, $\hat{A} = e(A)$, where A is any (compact) subset of ∂C.*

Since the crumpled m-cube C ($m = n+k$) satisfying the conclusions of Proposition (2.1) is a suspension (cf. [6]), it follows from Theorems (8B.6) and (8B.7) of [5] that the identity sewing of C is homeomorphic to S^m. Thus, the hypotheses of Proposition (2.2) are satisfied. Let $e: C \to Q$ denote the embedding given by Proposition (2.2). In this setting we have the following

(2.3) PROPOSITION. *The spaces $Q - \hat{\alpha}$, $S^m - \alpha$, $C - \alpha$, and $H^n - \{x\}$ are homotopy equivalent, where $\hat{\alpha} = e(\alpha)$ (H^n, C, S^m, α, and $m = n+k$ as in Proposition (2.1)).*

3. The method and the first example.

(3.1) *The method.* Our method is based on the following key result of Section 2:

For each homology n-sphere with $\pi_1 H^n \not\approx \{1\}$ and each integer $k > 1$, the arc α constructed in S^m ($m = n+k$) and the corresponding arc $\hat{\alpha}$ constructed in Q by applying Propositions (2.1) and (2.3) have the property that $S^m - \alpha$, $Q-\hat{\alpha}$, and $H^n - \{x\}$ are homotopy equivalent. Therefore, the computation of $\pi_i(S^m - \alpha)$ or $\pi_i(Q-\hat{\alpha})$, $i > 0$, is reduced to the computation of $\pi_i(H^n - \{x\})$; this can be often done for some carefully chosen H^n. Here, the basic idea is that if one has a good understanding of the universal cover $\tilde{H}^n$ of H^n, then one is bound to succeed in computing $\pi_i(H^n - \{x\})$ for some (all) $i > 1$ (assuming $\pi_1 H^n \approx \pi_1(H^n - \{x\})$ is already known). The following examples demonstrate this technique.

(3.2) *The first example.* Let H denote the Poincaré homology 3-sphere [13]. It is well known that $\pi_1 H$ has order 120 and a presentation

$$\langle x, y\colon x^3 = y^5 = (yx)^2\rangle;$$

this group is known as the binary icosahedral group. The universal cover $\tilde{H}$ of H is S^3. For each $m > 4$, we apply Propositions (2.1) and (2.3) to obtain an arc α in S^m and the corresponding arc $\hat{\alpha}$ in Q satisfying the following:

(3.2.1) $\pi_1(S^m - \alpha) \approx \pi_1(Q-\hat{\alpha}) \approx \pi_1(H - \{x\}) \approx \pi_1 H$;

(3.2.2) $\pi_i(S^3 - \alpha) \approx \pi_i(Q-\hat{\alpha}) \approx \pi_i(H - \{x\})$ for $i > 1$;

(3.2.3) $\pi_i(H - \{x\}) \approx \pi_i(S^3 - p^{-1}(x))$ for $i > 1$, where $p\colon \tilde{H} \to H$ denotes the covering projection.

In this setting, observe that $p^{-1}(x)$ is a set consisting of 120 points. Therefore, $\tilde{H} - p^{-1}(x)$ is up to homotopy a wedge B of 119 copies of the 2-sphere. Observe that:

(3.2.4) $\pi_i B \approx \pi_i(H - \{x\})$ for $i > 1$;

(3.2.5) $\pi_i B \approx \pi_i(S^m - \alpha) \approx \pi_i(Q-\hat{\alpha})$ for $i > 1$.

The group $\pi_i B$ ($i > 1$) can be expressed as a direct sum of the homotopy groups of spheres of suitable dimensions. This is a well-known result of Hilton [9].

(3.3) *The first example: computations.* The following is intended as a brief introduction to "the method of Hilton" (see Corollary (4.10) of [9]); consequently, we compute a few groups $\pi_i B$ as a sample.

(3.3.1) The *Möbius function* μ is defined as follows: $\mu(1) = 1$, $\mu(n) = 0$ unless n is square free, and $\mu(n) = (-1)^k$ when n is the product of k distinct primes (of course, n is a positive integer).

(3.3.2) For each integer $k > 0$, a *basic product function* f is defined by the rule

$$f(w) = \frac{1}{w} \sum_{d|w} \mu(d)(k)^{w/d}.$$

(3.3.3) Suppose T is a wedge of k two-spheres. A formula of Hilton [9] can now be stated as follows:

$$\pi_i T \approx (\bigoplus_{f(1)} \pi_i S^2) \oplus (\bigoplus_{f(2)} \pi_i S^3) \oplus \ldots \oplus (\bigoplus_{f(i-1)} \pi_i S^i),$$

where $\bigoplus_{f(1)} \pi_i S^2$ means the direct sum of $f(1)$ copies of $\pi_i S^2$ and similarly for others.

(3.3.4) A table for values of f with $k = 119$:

w	1	2	3	4	...
$f(w)$	119	7021	561680	50129940	...

(3.3.5) Each of the groups $\pi_4 S^2$, $\pi_5 S^2$, $\pi_4 S^3$, $\pi_5 S^3$, or $\pi_5 S^4$ is isomorphic to the cyclic group Z_2 of order 2.

By careful combining the discussions given above, we have proved the following

(3.4) THEOREM. *For each $m > 4$, there exist an arc α in S^m and an arc $\hat{\alpha}$ in Q such that*

$$\pi_1 (S^m - \alpha) \approx \pi_1 (Q - \hat{\alpha}) \approx \pi_1 H,$$

$$\pi_i (S^m - \alpha) \approx \pi_i (Q - \hat{\alpha}) \approx \pi_i B \quad \textit{for } i > 1.$$

For instance, we have the following specific calculations:

(a) $\pi_2 (S^m - \alpha) \approx \pi_2 (Q - \hat{\alpha}) \approx \pi_2 B \approx \bigoplus_{119} Z$;

(b) $\pi_3 (S^m - \alpha) \approx \pi_3 (Q - \hat{\alpha}) \approx \pi_3 B \approx \bigoplus_{7140} Z$;

(c) $\pi_4 (S^m - \alpha) \approx \pi_4 (Q - \hat{\alpha}) \approx \pi_4 B \approx (\bigoplus_{7140} Z_2) \oplus (\bigoplus_{561680} Z)$;

(d) $\pi_5 (S^m - \alpha) \approx \pi_5 (Q - \hat{\alpha}) \approx \pi_5 B \approx (\bigoplus_{568810} Z_2) \oplus (\bigoplus_{50129940} Z)$.

Here, Z denotes the infinite cyclic group (recall: B is a wedge of 119 copies of S^2, H is the Poincaré homology 3-sphere, and Z_2 is the cyclic group of order 2).

4. Additional examples.

(4.1) *Arcs in S^m or Q by aspherical homology spheres.* Suppose $H = H^n$ is an aspherical homology n-sphere with $\pi_1 H \not\approx \{1\}$. Since H is a $K(\pi, 1)$, its universal cover $\tilde{H}$ is contractible. Consider $H - \{x\}$ and $\tilde{H} - p^{-1}(x)$, where $p\colon \tilde{H} \to H$ denotes the covering projection. Note that $p^{-1}(x)$ is a discrete subset of $\tilde{H}$ having the cardinality of $\pi_1 H$ (which is countable infinite). Let $x_1, x_2, \ldots$ be an enumeration of points in $p^{-1}(x)$ without repetitions. Choose a closed n-cell C in H containing x in its interior such that $p^{-1}(C)$ is a disjoint union of n-cells C_1, C_2, ... with x_i in Int C_i for $i \geqslant 1$. Let A_1 denote a polyhedral arc which meets $p^{-1}(C)$ in its end-points and joins a point in ∂C_1 to a point in ∂C_2. Suppose the arcs $A_1, \ldots, A_{k-1}$ have been defined. Define a polyhedral arc A_k which meets $p^{-1}(C)$ in its end-points and joins a point in ∂C_k to a point in ∂C_{k+1} such that A_k lies in the complement of an open set containing the union of arcs $A_1, \ldots, A_{k-1}$. Let N denote the union of C_1, A_1, C_2, A_2, ... Clearly, N is a strong deformation retract of $\tilde{H}$ (observe that the inclusion $N \to \tilde{H}$ is a homotopy equivalence). Thus, $N - p^{-1}(x)$ is a strong deformation retract of $\tilde{H} - p^{-1}(x)$. For each $i \geqslant 1$, deform $C_i - \{x_i\}$ into ∂C_i. This means that $N - p^{-1}(x)$ has the homotopy type of an infinite wedge W of $(n-1)$-spheres S_1, S_2, ...; consider W as the direct limit of the system

$$S\colon\ S_1 \to S_2 \vee S_2 \to (S_1 \vee S_2) \vee S_3 \to \ldots$$

whose bonds are the obvious inclusions. For each $i > 1$, we compute $\pi_i W$ as the direct limit $\varinjlim \pi_i S$. The computations of $\varinjlim \pi_i S$ $(i > 1)$ can be completed by using the results of Hilton [9] and the fact that each bond in the direct system $\pi_i S$ embeds its domain in its range as a direct summand (see [9]). Observe that

$$\pi_2 W \approx \pi_3 W \approx \ldots \approx \pi_{n-2} W \approx \{1\}$$

and

$$\pi_{n-1} W \approx \bigoplus_{\infty} Z$$

(this denotes the direct sum of countable infinite copies of Z), since each S_i $(i > 0)$ is an $(n-1)$-sphere. An enterprising reader may compute $\pi_n W$, $\pi_{n+1} W$, ...

The following theorem summarizes our discussions given above:

(4.1.1) THEOREM. *For each aspherical homology n-sphere* H^n *with* $\pi_1 H^n \not\approx \{1\}$ *and an integer* $k > 1$, *there exist an arc* α *in* S^m $(m = n+k)$ *and the corresponding arc* $\hat{\alpha}$ *in* Q *such that*:

(a) $\pi_1(S^m - \alpha) \approx \pi_1(Q - \hat{\alpha}) \approx \pi_1 H^n$;

(b) $\pi_i(S^m - \alpha) \approx \pi_i(Q - \hat{\alpha}) \approx \{1\}$ *when* $1 < i < n-1$;

(c) $\pi_i(S^m - \alpha) \approx \pi_i(Q - \hat{\alpha}) \approx \bigoplus_{\infty} Z$ *when* $i = n-1$;

(d) $\pi_i(S^m - \alpha) \approx \pi_i(Q - \hat{\alpha}) \approx \pi_i W$ *when* $i > n-1$.

(Recall: W denotes the wedge of $(n-1)$-spheres S_1, S_2, ...)

We now provide examples of homology spheres which are $K(\pi, 1)$ spaces.

(4.2) *Brieskorn homology 3-spheres.* Suppose p, q, and r are integers greater than 1. Let C^3 denote the 3-fold product of the complex plane C.

A *Brieskorn manifold* $M = M(p, q, r)$ is the set of all points (z_1, z_2, z_3) of C^3 satisfying

$$z_1^p + z_2^q + z_3^r = 0 \quad \text{and} \quad |z_1|^2 + |z_2|^2 + |z_3|^2 = 1.$$

The following is a well-known result of Brieskorn (cf. [10]):

(4.2.1) *If p, q, and r are pairwise relatively prime, then M is a homology 3-sphere called a Brieskorn homology 3-sphere.*

The following results are also well known:

(4.2.2) *$M(2, 3, 5)$ is homeomorphic to the Poincaré homology 3-sphere; this is the only Brieskorn homology 3-sphere with finite fundamental group.*

(4.2.3) *A Brieskorn homology 3-sphere with infinite fundamental group is a $K(\pi, 1)$.*

(4.2.4) *Two Brieskorn homology 3-spheres with infinite fundamental groups are homeomorphic if and only if their fundamental groups are isomorphic.*

A presentation of the fundamental group of a Brieskorn (homology 3-sphere) manifold is given in [11].

Our main reference for these results is [11]; see also [10] and [12] where additional references may also be found.

(4.3) THEOREM. *For each Brieskorn homology 3-sphere H with infinite $\pi_1 H$ and integer $k > 1$, there exist arcs α in S^m $(m = 3+k)$ and $\hat{\alpha}$ in Q such that*

$$\pi_1(S^m - \alpha) \approx \pi_1(Q - \hat{\alpha}) \approx \pi_1 H,$$

$$\pi_i(S^m - \alpha) \approx \pi_i(Q - \hat{\alpha}) \approx \pi_i W \quad \textit{for all } i > 1.$$

For instance, we have the following specific calculations:

(a) $\pi_2(S^m - \alpha) \approx \pi_2(Q - \hat{\alpha}) \approx \pi_2 W \approx \bigoplus\limits_{\infty} Z$;

(b) $\pi_3(S^m - \alpha) \approx \pi_3(Q - \hat{\alpha}) \approx \pi_3 W \approx \bigoplus\limits_{\infty} Z$;

(c) $\pi_4(S^m - \alpha) \approx \pi_4(Q - \hat{\alpha}) \approx \pi_4 W \approx (\bigoplus\limits_{\infty} Z_2) \oplus (\bigoplus\limits_{\infty} Z)$;

(d) $\pi_5(S^m - \alpha) \approx \pi_5(Q - \hat{\alpha}) \approx \pi_5 W \approx (\bigoplus\limits_{\infty} Z_2) \oplus (\bigoplus\limits_{\infty} Z)$.

The space W is a wedge of countable infinite copies of the 2-sphere; this is discussed in (4.1).

Proof. Compute $\pi_i T$ $(i > 1)$ for a wedge T of n two-spheres; use (3.3.3). Our proof is completed by observing that as n goes to infinity (i.e., if we take the direct limit), the values of $f(w)$ as in (3.3.4) also go to infinity.

(4.4) Remarks. Theorems (3.4) and (4.3) exhaust all the Brieskorn homology 3-spheres. Recall that the Poincaré homology 3-sphere is the Brieskorn manifold $M(2, 3, 5)$; this is used in Theorem (3.4). The remaining

Brieskorn homology 3-spheres have infinite fundamental groups; these are used in Theorem (4.3). It is easy to see that there are countable infinite topologically distinct Brieskorn homology 3-spheres (see [10]–[12]). Observe that if one suitably varies m or α in Theorem (4.3), the groups $\pi_i(S^m-\alpha)$ or $\pi_i(Q-\hat{\alpha})$ do not change up to isomorphism for $i>1$.

(4.5) *Additional results.* Here is another method of producing homology 3-spheres which are $K(\pi, 1)$ spaces. Let M_1 and M_2 denote complements of *non-trivial smooth knots* in S^3. Construct a 3-manifold M from M_1 and M_2 by identifying their boundaries by a diffeomorphism which takes the longitude of one to the meridian of the other (see [13], p. 251). The following facts are easy to prove:

(a) *M is a homology 3-sphere*;

(b) *M is a $K(\pi, 1)$*;

(c) *$\pi_1 M$ is an amalgamated free product of $\pi_1 M_1$ and $\pi_1 M_2$ along $Z\oplus Z$.*

Clearly, if α or $\hat{\alpha}$ are arcs constructed in S^m with $m>4$ or Q, respectively, by utilizing M, then

$$\pi_1(S^m-\alpha)\approx\pi_1(Q-\hat{\alpha})\approx\pi_1 M$$

and

$$\pi_i(S^m-\alpha)\approx\pi_i(Q-\hat{\alpha})\approx\pi_i W \quad \text{for } i>1$$

(W is the wedge of countable infinite 2-spheres as above).

(4.6) Remark. For each $m>2$, there exist arcs α in S^m and the corresponding arcs $\hat{\alpha}$ in Q such that $S^m-\alpha$ and $Q-\hat{\alpha}$ are $K(\pi, 1)$ spaces with

$$\pi_1(S^m-\alpha)\approx\pi_1(Q-\hat{\alpha})\not\approx\{1\}.$$

This follows easily by suitably applying a "suspension method" to a suitable arc in S^3; see [2] and [6] for details.

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