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"ZERO-TWO" LAW FOR CONSERVATIVE MARKOV OPERATORS

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In [6] Ornstein and Sucheston considered the limit behaviour of $T^n(I-T^k)$ and proved the so-called "zero–two" law for Markov operators T on L^∞ spaces. Foguel [3] extended and simplified the Ornstein–Sucheston result. In this note we obtain similar results for certain Markov operators acting on the more general space $C(X)$ of continuous functions over a compact Hausdorff space X. We shall appeal to a few facts on conservative operators as presented in [2] and on quasi-compact operators as presented in [7]. It is not hard to see that the metrizability assumption in all results quoted from [2] and [4] is unnecessary.

Let X be a compact Hausdorff space. By a *Markov operator* we mean a linear operator

$$T\colon C(X) \to C(X)$$

such that $f \geqslant 0 \Rightarrow Tf \geqslant 0$ and $T1 = 1$. It is well known (see e.g. [8]) that for every Markov operator T there exists a unique family of probability Radon measures $P(x, \cdot)$, $x \in X$, on X such that: (a) for every Borel set A the mapping $x \to P(x, A)$ is Borel measurable, (b) for every $f \in C(X)$ $Tf = \int f(y) P(\cdot, dy)$. In fact we have $P(x, \cdot) = T^* \varepsilon_x(\cdot)$. By $B(X)$ we denote the bounded Borel functions on X. Using (b) the Markov operator T can be canonically extended to an operator $T\colon B(X) \to B(X)$. If T takes $B(X)$ into $C(X)$ then T is called *strong Feller*.

LEMMA 1. *Let $T\colon C(X) \to B(X)$ be a positive linear operator. If Tf is upper semicontinuous (u.s.c.) for every nonnegative continuous function f then there exists a family of probability Radon measures $P(x, \cdot)$, $x \in X$, on X such that* (a) *and* (b) *hold.*

Proof. It is enough to show that for every Borel set A the mapping $x \to T^* \varepsilon_x(A)$ is Borel measurable. The class of Borel sets for which this holds contains all closed sets, is closed under taking unions of monotone sequences and under relative complements. Now the assertion follows from [7], Chapter 1, Theorem 3.2.

It should be noted that if X is metrizable then the indicator of every closed set is the pointwise infimum of a sequence of continuous functions. Thus in Lemma 1 the assumption of upper semicontinuity can be omitted.

Now let T_1, T_2 be two positive operators on $C(X)$. An essential step in the proof of the "zero-two" law for Markov operators on L^∞ is the definition of the lattice theoretic infimum $S_n = T^{n+k} \wedge T^k$ (see [3]). Here by $T_1 \wedge T_2$ we shall denote the positive linear operator acting from $C(X)$ to $B(X)$ defined for $f \geqslant 0$ by

$$T_1 \wedge T_2 f(x) = \inf\{T_1 g(x) + T_2(f-g)(x):\ 0 \leqslant g \leqslant f,\ g \in C(X)\}.$$

From Lemma 1 the operator $T_1 \wedge T_2$ can be canonically extended to an operator from $B(X)$ to $B(X)$. The following "zero-two" law corresponds to Theorem I in [3]. Its proof is now a slight modification of the proof in [3] and is omitted.

THEOREM 1. *Let T be a Markov operator on $C(X)$ and let $k \geqslant 0$. If $\|T^{m+k} - T^m\| < 2$ for some $m \geqslant 0$ then $\|T^{n+k} - T^n\| \to 0$ as $n \to \infty$.*

In the sequel we shall obtain more complete information on the "zero-two" behaviour of iterates for a narrower class of operators. Recall that a linear operator T: $C(X) \to C(X)$ is compact iff the mapping $x \to T^* \varepsilon_x$ is norm continuous (see [7], Proposition 5.9). In particular every compact Markov operator is strong Feller. As in [2] let A_T be the family of nonnegative l.s.c. functions f such that $Tf \leqslant f$ and $T^n f \to 0$ pointwise. By the *Foguel boundary* F_T of T we mean the intersection of all zero-sets $\{x: f(x) = 0\}$ for the functions f in A_T. If $F_T = X$ we say that T is *conservative*. Let $P_T(X)$ denote the set of all T^*-invariant probability measures. By the *center* M_T of T we mean the closure of the union of the supports of all measures in $P_T(X)$.

The Markov operator T is said to be *quasi-compact* if there exists a sequence R_n of compact operators on $C(X)$ such that $\|T^n - R_n\| \to 0$ (our definition differs slightly from that in [7] and is narrower since the domain of our operators is $C(X)$). For a quasi-compact Markov operator T the condition $M_T = X$ is equivalent to the conservativeness of T. Indeed, it is well known that $M_T \subseteq F_T$ for all Markov operators, so we only have to show that if T is conservative then $M_T = X$. From the inequality $T1_{M_T} \geqslant 1_{M_T}$ it is easy to see that the pointwise limit of $T^n 1_{M_T}$ exists and, by [5], Theorem 1, belongs to $C(X)$. Thus if $\lim T^n 1_{M_T} \neq 1_{M_T}$ then we have $T^n 1_{M_T} \neq 1_{M_T}$ on a set of second category for some n. But this is impossible from the conservativeness of T. Thus $T1_{M_T} = 1_{M_T} \in C(X)$ and $M_T = X$ since M_T is clopen.

Remark 1. Let P be a stochastic operator (i.e. $P \geqslant 0$ and $P^* 1 = 1$) on an $L^1(m)$ where m is a probability measure. The stochastic operator P is called *conservative* if $P^*f \leqslant f \Rightarrow P^*f = f$ for $f \in L^\infty(m)$ (see e.g. [1]). Since $L^\infty(m)$ is isomorphic to $C(X)$ for some compact Hausdorff space X, there exists a Markov operator $\tilde{P}$: $C(X) \to C(X)$ naturally corresponding to P^*. It is not hard to observe that P is conservative if and only if $\tilde{P}$ is conservative in the sense of our previous definition. Indeed, let $\tilde{P}$ be a conservative Markov operator on $C(X)$, and $P^*f \leqslant f$. Thus $\tilde{P}\tilde{f} \leqslant \tilde{f}$ (with $\tilde{f}$ the image of f in $C(X)$) and by conservativeness of $\tilde{P}$ we have $\tilde{P}\tilde{f} = \tilde{f}$, so $P^*f = f$. Now suppose that P is a conservative stochastic operator but $\tilde{P}$ is not. Then, by [4], Theorem 1.1, there exists a nonvoid, clopen set $\tilde{U} \subset X$ such that $\sum_{n=0} \tilde{P}^n 1_{\tilde{U}}(x) \leqslant L < \infty$ for some L and every $x \in X$. Thus we get $\sum_{n=0} P^{*n} 1_U \leqslant L$ in $L^\infty(m)$ for some set U $(m(U) > 0)$, but this is impossible from conservativeness of T.

The asymptotic behaviour of iterates of quasi-compact operators is known (see [7], Chapter 6). The following lemma is a topological version of Theorem 3.7, p. 176, in [7] and its proof is modelled on the proof of the quoted theorem. It is very important for us that the following lemma holds in a nonseparable case too (the separability assumption in Theorem 3.7 in [7] is unnecessary).

LEMMA 2. *Let T be a quasi-compact, conservative Markov operator on $C(X)$. Then there exist a partition of X into clopen sets $E_{\varrho,\delta}$ and probability measures $m_{\varrho,\delta}$ with supports $E_{\varrho,\delta}$ ($\varrho = 1, \ldots, r$, $\delta = 1, \ldots, d_\varrho$) such that if d denotes the least common multiple of the d_ϱ's then for every k the operators T^{nd+k} converge to $\sum_{\varrho=1}^{r} \sum_{\delta=1}^{d_\varrho} 1_{E_{\varrho,\delta-k}} \otimes m_{\varrho,\delta}$ in the norm topology* (the second subscripts taken modulo d_ϱ, see Proposition 3.5 of [7]).

Proof. By [7] the iterates T^{nd} norm converge to a projection

$$S = \sum_{\varrho=1}^{r} \sum_{\delta=1}^{d_\varrho} U_{\varrho,\delta} \otimes m_{\varrho,\delta}$$

where $U_{\varrho,\delta} \in B(X)$ and $m_{\varrho,\delta} \in P(X)$. Since $SU_{\varrho,\delta} = U_{\varrho,\delta}$ (see [7]) and S is a finite rank (hence strong Feller) operator, $U_{\varrho,\delta} \in C(X)$. Let $E_{\varrho,\delta} = \{x\colon U_{\varrho,\delta}(x) = 1\}$. By the proof of Theorem 3.7

$$1_{E_{\varrho,\delta}} \leqslant T^d 1_{E_{\varrho,\delta}} \leqslant \ldots \leqslant \lim_n T^{nd} 1_{E_{\varrho,\delta}} = S 1_{E_{\varrho,\delta}} \in C(X).$$

By the conservativeness of T and the closedness of $E_{\varrho,\delta}$ we have

$$1_{E_{\varrho,\delta}} = T^d 1_{E_{\varrho,\delta}} = \ldots = S 1_{E_{\varrho,\delta}},$$

so $E_{\varrho,\delta}$ is clopen. In the same way using $T1_{E_{\varrho,(\delta+1)}} \geqslant 1_{E_{\varrho,\delta}}$ we can show that $T1_{E_{\varrho,(\delta+1)}} = 1_{E_{\varrho,\delta}}$ for every $\delta = 1, \ldots, d_\varrho$. We shall show that $\bigcup_{\varrho,\delta} E_{\varrho,\delta} = X$.

Since the set $X_0 = X - \bigcup_{\varrho=1}^{r} \bigcup_{\delta=1}^{d_\varrho} E_{\varrho,\delta}$ is clopen and T-invariant, we have $S1_{X_0} = 1_{X_0}$. On the other hand

$$S1_{X_0} = (\sum_{\varrho} \sum_{\delta} U_{\varrho,\delta} \otimes m_{\varrho,\delta}) 1_{X_0} = 0,$$

therefore $X_0 = \emptyset$. Since

$$U_{\varrho,\delta} \geqslant 1_{E_{\varrho,\delta}} \quad \text{and} \quad \sum_{\varrho=1}^{r} \sum_{\delta=1}^{d_\varrho} U_{\varrho,\delta} = 1$$

(see [7]), $U_{\varrho,\delta} = 1_{E_{\varrho,\delta}}$ and the lemma is proved.

Now for quasi-compact conservative Markov operators we have the following sharper version of Theorem 1.

THEOREM 2. *Let T be a quasi-compact conservative Markov operator on $C(X)$ and let $d \geqslant 1$. Then there exists a partition of X into two clopen T-invariant sets X_0 and X_2 such that T^{nd} converge on $C(X_0)$ in the norm topology and $\|(T^n(I-T^d))^* \varepsilon_x\| = 2$ for every $x \in X_2$.*

Proof. Let

$$J = \{\varrho: 1 \leqslant \varrho \leqslant r \text{ and } d_\varrho \text{ divides } d\},$$

$$X_0 = \bigcup_{\varrho \in J} \bigcup_{\delta=1}^{d_\varrho} E_{\varrho,\delta} \quad \text{and} \quad X_2 = X \setminus X_0.$$

Applying Lemma 2 to $T|_{C(X_0)}$ we can easily show that T^{nd} converge to $\sum_{\varrho \in J} \sum_{\delta=1}^{d_\varrho} 1_{E_{\varrho,\delta}} \otimes m_{\varrho,\delta}$ in the norm topology on $C(X_0)$. Now let $x \in X_2$. Thus $x \in E_{\varrho,\delta}$ where $1 \leqslant \varrho \leqslant r$, $\varrho \notin J$ and $1 \leqslant \delta \leqslant d_\delta$. From Lemma 2 we get

$$\operatorname{supp} T^{*j} \varepsilon_x \subset E_{\varrho,\delta+j}.$$

Thus

$$\operatorname{supp} T^{*nd} \varepsilon_x \cap \operatorname{supp} T^{*(n+1)d} \varepsilon_x \subset E_{\varrho,(\delta+nd)} \cap E_{\varrho,(\delta+(n+1)d)} = \emptyset$$

because $\delta + nd \not\equiv \delta + (n+1)d \pmod{d_\varrho}$.

Remark 2. The assertion of Theorem 2 holds for every strong Feller conservative operator because T^2 is then compact by [7], Theorem 5.10, p. 35 (the metrizability assumption in Theorem 5.10 causes no loss of generality since if the restriction to every separable subspace is compact then the operator is compact). Without any compactness assumption on T^m the situation is less clear:

Example. Let $X = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$. Define

$$Tf(x) = f(g(x))$$

where $g(0) = 0$ and

$$g(1/n) = \begin{cases} 1/(n+1), & n \text{ odd}, \\ 1/(n-1), & n \text{ even}. \end{cases}$$

For this Markov operator the assertion of Theorem 2 does not hold for $d = 1$.

Remark 3. If T is a quasi-compact Markov operator then (by Rosenblatt [8], Theorem 6, and from Lemma 2) the iterates converge in the norm if and only if 1 is unique peripheral eigenvalue of T.

Remark 4. If T is a quasi-compact, conservative Markov operator on $C(X)$ where X is a compact Hausdorff connected space, then from Lemma 2 we infer that T^n converge in the norm to $1 \otimes m$ where m is a unique T^*-invariant probability measure.

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