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LOCALLY CONNECTED IMAGES OF ORDERED COMPACTA*

BY

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In 1963 in [4] Mardešić and Papić raised the question "Is every locally connected continuum M which is the continuous image of an ordered compactum K also the continuous image of an arc (nondegenerate ordered continuum)?" The main purpose of this paper is to show that if (1) P denotes the set of all points x of M (as above) such that every neighborhood of x contains a non-metrizable subcontinuum of M, and (2) G denotes the decomposition of M into components of P and points of $M-P$, then (3) M/G is the continuous image of an arc.

There are a number of other partial results connected with the question above. Mardešić shows in [3] and Cornette and Lehman show in [2] that there is a nondegenerate locally connected continuum which is not pathwise connected by continuous images of arcs. In [5] Mardešić shows that if X is the continuous image of an ordered compactum K and G is an open F_σ-set in X such that $\bar{G}$ is connected, then $\mathrm{Bd}(G)$ is metrizable. In [7] Pearson shows that if H is a continuum in which each pair of points is separated by a finite set, then H is the continuous image of an arc. L. E. Ward Jr. shows in [13] that if a continuum X can be approximated by finite trees, then X is the continuous image of an arc. In [8] Simone shows that if K is a paraseparable continuum containing no nondegenerate metric subcontinuum, then K is netlike if and only if K is the continuous image of some ordered compactum. Treybig shows in [11] that if the continuum M is the continuous image of an ordered compactum and x, y are points of M contained in no metric subcontinuum of M, then x, y are separated in M by a finite set. In [15] Young shows that if L is the "long arc" then $L\times[0, 1]$ is not the continuous image of an ordered compactum.

In this paper all topological spaces are Hausdorff, and all ordered topological spaces are provided with the interval topology. A *compactum* is a

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compact Hausdorff space, and an *arc* is a nondegenerate ordered continuum. *Local connectivity* and *upper semi-continuous collections* are defined in [14]. A *continuum* is a connected compactum, and a continuum X is a *tree* if each pair of points of X is separated in X by a point of X. A continuum X is *locally peripherally finite* if each point of X has arbitrarily small neighborhoods with finite boundaries.

Throughout this paper let $f\colon K \to M$ be a continuous mapping of the ordered compactum K onto the locally connected continuum M. By Lemma 4 of [9] we may assume without loss of generality that (1) f maps no closed proper subset of K onto M and (2) if x and y are points of K such that $f(x) = f(y)$, then there is a point z of K between x, y such $f(z) \neq f(x)$. Let A', B' denote, respectively, the first and last points of K. Given subintervals I, J of K such that $f(I)$ intersects $f(J)$, let x_{IJ}, x'_{IJ} ($x = a, b, c, d$) denote points of I, J respectively, such that (1) $f(x_{IJ}) = f(x'_{IJ})$ for $x = a, b, c, d$, and (2) if $t \in I$, $t' \in J$ and $f(t) = f(t')$, then $a_{IJ} \leqslant t \leqslant b_{IJ}$ and $c'_{IJ} \leqslant t' \leqslant d'_{IJ}$. We let P, G be as in paragraph one.

At this stage it might help to see what the set P looks like in some examples. Let $I_1, I_2, \ldots$ denote a simple sequence of copies of [0, 1], and let L_1 denote $I_1 \times I_2 \times \ldots$ with the topology induced by the lexicographic order (see [1]). Let L_2 denote the "long interval". In the plane let M' denote the union of the intervals $[(0, y), (1, y)]$, $y = 0, 1, \frac{1}{2}, \frac{1}{4}, \ldots$, and the intervals $[(q/2^p, 0), (q/2^p, 1/2^p)]$, $p = 0, 1, 2, \ldots$, and q an integer in $[0, 2^p]$.

Example i ($i = 1, 2$). Form M^i from M' by replacing each maximal vertical interval whose interior is open in M' by a copy of L_i. The topology is determined in such a way that there is a natural monotone map from M^i onto M'. If $M = M^2$ the set P has a dense set of degenerate components, and if $M = M^1$ the set P is a continuum. It may be seen with the aid of the Hahn–Mazurkiewicz Theorem applied to M' that each M^i is the continuous image of an arc.

THEOREM 1. *G is an upper semi-continuous decomposition of M into continua such that M/G is a locally connected continuum which is the continuous image of an ordered compactum.*

Proof. The last part of the theorem follows by considering $K \xrightarrow{f} M \xrightarrow{\varphi} M/G$, where φ is the natural map. The first part of the proof follows from well-known properties of upper semi-continuous collections (see [14]).

THEOREM 2. *Suppose the subcontinuum L of M fails to be locally connected at the point P of L and U is an open set containing P. Then there exist* (1) *open sets R_1, R_2, R_3 such that $P \in R_1 \subset \bar{R}_1 \subset R_2 \subset \bar{R}_2 \subset R_3 \subset \bar{R}_3 \subset U$,* (2) *components $L_1, L_2, \ldots$ of $L \cap \bar{R}_3$, and* (3) *sequences of points $u_1, u_2, \ldots$; $v_1, v_2, \ldots$; $w_1, w_2, \ldots$ such that* (a) *for each i $u_i \in L_i \cap \mathrm{Bd}(R_1)$, $v_i \in L_i \cap \mathrm{Bd}(R_2)$, and $w_i \in L_i \cap \mathrm{Bd}(R_3)$,* (b) *$u_1, u_2, \ldots$ converges to a point u; $v_1, v_2, \ldots$ converges to a point v; $w_1, w_2, \ldots$ converges to a point w; and* (c) *the limiting set L'* (see

[6]) *of* L_1, L_2, ... *contains* u, v, w *and is a connected subset of a component* C *of* $L \cap \bar{R}_3$, *where* C, L_1, L_2, ... *are distinct.*

Proof. The proof follows by modifying the proofs of Theorem 11, page 90, and Theorem 58, page 23, of [6] and also noting the fact that a modification of Lemma 6 of [9] yields that every infinite sequence in M has a convergent subsequence.

THEOREM 3. *Suppose* C *is a locally connected metric continuum,* H *a totally disconnected closed subset of* C, *and* R *a countable subset of* C *such that every limit point of* R *is a point of* H. *Then, there is a tree* T *containing* R *and lying in* C.

Proof. Let G_1, G_2, ... denote a collection of finite covers of C by connected open sets such that (1) each G_{n+1} refines G_n and (2) the diameter of each element of G_n is less than $1/n$. Let $x_0 \in C - R$.

First let $g_1, \ldots, g_j$ denote all the elements of G_1 which contain a point of R. There is an arc $x_0 p_1$ from x_0 to a point p_1 of R in g_1. Next let j_2 denote the least index t so that $g_t \cap x_0 p_1$ is void. There is an arc $q_2 p_2$ from a point q_2 of $x_0 p_1$ to a point p_2 of $R \cap g_{j_2}$ so that $x_0 p_1 \cap q_2 p_2 = \{q_2\}$. Let j_3 denote the least index t so that $g_t \cap [x_0 p_1 \cup q_2 p_2]$ is void. There is an arc $q_3 p_3$ such that $p_3 \in g_{j_3} \cap R$ and $((x_0 p_1 \cup q_2 p_2) \cap q_3 p_3) = q_3$. Continuing in this way we find a tree T_1 such that (1) T_1 intersects each $g_i (1 \leqslant i \leqslant j)$ and (2) T_1 is the union of arcs $I_1, \ldots, I_{n_1}$ so that (a) $I_i \cap R$ is not void $(1 \leqslant i \leqslant n_1)$ and (b) if $1 < i$ then $I_i \cap [I_1 \cup \ldots \cup I_{i-1}]$ is an endpoint of I_i.

Now suppose T_1, $T_2, \ldots, T_n$ have been defined where (1) T_n is a finite tree intersecting every element of $G_1 \cup \ldots \cup G_n$ which contains a point of R, and (2) T_n is the union of an analogous collection of arcs as above. Let the elements of G_n which contain a point of R be labeled $h_1, h_2, \ldots, h_x$, and for each i $(1 \leqslant i \leqslant x)$ let $k_{i1}, k_{i2}, \ldots, k_{iy_i}$ denote the elements of G_{n+1} which lie in h_i and contain a point of R not in T_n. Now making changes only in h_1 we add to T_n to form a finite tree S_1 such that (1) $T_n \subset S_1$, (2) S_1 meets every element of $k_{11}, k_{12}, \ldots, k_{1y_1}$, and (3) S_1 is the union of such a collection of arcs as above. Analogously, we form S_2 to take care of $k_{21}, k_{22}, \ldots, k_{2y_2}$ while making additions only in h_2. We continue in this way to find $S_3, \ldots, S_x$ and let $T_{n+1} = S_x$. Note that every arc added to form T_{n+1} from T_n is a subset of some element of G_n. Finally we let $T' = \mathrm{Cl}(\bigcup_1^{\infty} T_n)$.

Clearly T' is connected and contains R. To show that T' is locally connected, first let $x \in T' - H$. There is a positive integer n such that the distance from x to H is greater than $1/n$ so let $N(x, 1/n)$ be the open neighborhood of x of radius $1/n$. Since $M - H$ contains no limit points of R, let $R \cap N(x, 1/2n) = \{r_1, \ldots, r_j\}$, and let m be an integer greater than $4n$ so that if g, g' are elements of G_m containing elements r, r' of C, respectively, where $r = r_i$ $(1 \leqslant i \leqslant j)$ and $r' = r_k$ $(k \neq i)$ or $r' \in C - N(x, 1/2n)$, then $\bar{g} \cap \bar{g}'$ is void.

If $1 \leqslant i \leqslant j$, then by the construction above $r_i \in T_{m+1}$, and if $g_1, g_2, \ldots, g_j, g$ are elements of G_t $(t \geqslant m+1)$ and $r_i \in g_i$ $(1 \leqslant i \leqslant j)$ and g contains some $r \in R - \{r_1, \ldots, r_j\}$ and a change is made in g to form T_{t+1}, then no change is made in $N(x, 1/4n)$. Thus, since T_{t+1} is locally connected at x and $T_{t+1} \cap N(x, 1/4n) = T' \cap N(x, 1/4n)$, T' is locally connected at x. T' is locally connected at every point, since if not, by Theorem 2, T' would fail to be locally connected at each point of some nondegenerate subcontinuum of T'. This is impossible since $T' \cap H$ is totally disconnected.

T' is the closure of the union of a countable collection of arcs $I_1, I_2, I_3, \ldots$ such that (1) each I_p contains a point of R, (2) each $I_{p+1} \cap (I_1 \cup \ldots \cup I_p)$ is a single point, and (3) if U is an open set containing H, then there is a positive integer N such that if $p > N$, then $I_p \subset U$.

Let $s_1, s_2, \ldots$ be a countable set dense in T' and such that if J is a simple closed curve in T', then there is an arc $s_{n_1} s_{n_2}$ lying in J, where $s_{n_1} s_{n_2}$ lies in a segment cd of some I_k where cd is open in T' and contains no point of R. Let V denote the set of all such arcs $s_{n_1} s_{n_2}$ and let the elements of V be labeled $v_1, v_2, \ldots$ Now let $W_1 = T' - \operatorname{seg} v_1$, where $\operatorname{seg} ab$ means $\operatorname{arc} ab - \{a, b\}$. If W_1 contains a simple closed curve, then let j_2 be the least integer t such that v_t lies in such a simple closed curve and $\operatorname{seg} v_t$ is open in W_1. Let $W_2 = W_1 - \operatorname{seg} v_{j_2}$. We continue the process to find $v_{j_3}, W_3, v_{j_4}, W_4, \ldots$ Let T denote the continuum $\bigcap_1^\infty W_i$.

Let $x \in T - H$ and U be an open set containing x such that $\bar{U} \subset C - H$. There is a positive integer N so that $n > N$ implies $I_n \subset M - \bar{U}$.

If v_{j_i} intersects U for infinitely many i, then the properties of T_N imply that there is an arc α such that $\operatorname{seg} \alpha$ is open in T_N, lies in U, and intersects two of the v_{j_i}. This means one of the v_{j_i} cannot lie on a simple closed curve in W_{j_i}.

If v_{j_i} does not intersect U for infinitely many i, then there is a positive integer n such that $T \cap U = T_n \cap U \subset U \subset \bar{U} \subset C - H$.

If T is not locally connected at x let $L_1, L_2, \ldots, L'$ be as in Theorem 2. This situation is impossible since every nondegenerate subcontinuum of T_n contains a segment which is open in T_n. Thus T is locally connected at each x in $T - H$. As above, since $T \cap H$ is totally disconnected, T is locally connected at every point.

THEOREM 4. *Suppose* (1) *P is totally disconnected and* (2) *$M - P$ is connected. Then there is a well ordered sequence $\{N_a : a \in A\}$ of separable compact subsets of M such that* (1) *if $a, b \in A$ and $a < b$, then $N_a \subset N_b$ and $N_a - P$ is a locally connected subset of* $\operatorname{Int}(N_b - P)$, (2) *if $a \in A$ and C is a component of $M - N_a$, then* $\operatorname{card}(\operatorname{Bd}(C)) \leqslant 2$, (3) $\operatorname{card} A \leqslant \aleph_1$, (4) $M - P \subset \bigcup_{a \in A} N_a$, *and* (5) *if $B = (a_1, a_2, \ldots)$ is a subsequence of A and C_{a_i} is a* *component of* $N_{a_i} - \mathrm{Cl}(\bigcup_{a<a_i} N_a)$ *for each* $a_i \in B$ *then* (i) *each* $\overline{C_{a_i}}$ *is a locally connected metric continuum,* (ii) *if* $P_{a_i} \in \overline{C_{a_i}}$ *for* $a_i \in B$ *and there is a point* R *of* M *such that if* R *is an element of the open set* U, *there is an index* i *so that if* $i \leqslant j$ *then* $P_{a_j} \in U$, *then for each such* U *there is an index* i *so that if* $i \leqslant j$ *then* $\overline{C_{a_i}} \subset U$, *and* (iii) *under the hypothesis of* (ii) *if* $R \notin P$, *then* $\{R\} \cup (\bigcup_{a_i \in B} \overline{C_{a_i}})$ *is locally peripherally finite at* R.

Proof. Let H denote the set of all countable subsets Q of $K = [A', B']$ such that $A', B' \in Q$ and $3 \leqslant \operatorname{card} f(Q)$.

LEMMA 4A. *Suppose that* $Q \in H$ *and* $A_1, A_2, \ldots$ *and* $B_1, B_2, \ldots$ *are orderings of the elements of* Q *such that* $A_1 = B_1 = A'$, $A_2 = B_2 = B'$, *and if* $i \neq j$ *then* $A_i \neq A_j$ *and* $B_i \neq B_j$. *Let* $X_1, X_2, \ldots$ *and* $Y_1, Y_2, \ldots$ *denote sequences of finite subsets of* K *such that* (1) $X_1 = \{A_1, A_2, A_3\}$, $Y_1 = \{B_1, B_2, B_3\}$, (2) $X_p \subset X_{p+1}$ *and* $Y_p \subset Y_{p+1}$ *for* $p = 1, 2, \ldots$, (3) *if* X_n (*resp.* Y_n) *has been defined, then* X_{n+1} (Y_{n+1}) *is the smallest set* T *such that* (i) $X_n \subset T$ ($Y_n \subset T$), (ii) $A_{n+1} \in T$ ($B_{n+1} \in T$), *and* (4) *if each of* p, q, r, s *is an element of* X_n (Y_n) *and* $I = [p, q]$, $J = [r, s]$ *and* $f(I)$ *intersects* $f(J)$, *then* T *contains* x_{IJ}, x'_{IJ} *for* $x = a, b, c, d$. *Then* $\bigcup X_n = \bigcup Y_n$.

Proof. Clearly $X_1 = \{A_1, A_2, A_3\} \subset \bigcup Y_n$ since $A_3 = B_j$ for some j and $X_1 \subset Y_j$. Now suppose $X_p \subset Y_q$. Since $A_{p+1} = B_s$ for some integer s, then $X_{p+1} \subset Y_{q+s}$. By induction $\bigcup X_n \subset \bigcup Y_n$, and likewise $\bigcup Y_n \subset \bigcup X_n$.

Definition. Given Q in H define $L(Q)$ to be $\overline{\bigcup X_n}$, where the X_n's are defined in Lemma 4A. Define $F(Q)$ to be $f(\overline{L(Q)})$.

LEMMA 4B. *If* $Q \in H$ *and* C *is a component of* $M - F(Q)$, *then* $\mathrm{Bd}(C) \subset F(Q)$ *and* $\operatorname{card} \mathrm{Bd}(C) \leqslant 2$.

Proof. Let E denote the set of all subsets of K which are maximal relative to the property of being convex subsets of $K - \overline{L(Q)}$. Let $c_0 \in C$ and suppose the element (r, s) of E intersects $f^{-1}(c_0)$. Let $H' = \{g \in E|$ there exist elements $g_0 = (r, s), g_1, \ldots, g_n = g$ of E such that $f(g_p)$ intersects $f(g_{p+1})$ for $p = 0, \ldots, n-1\}$. Since some $\operatorname{card}(f(X_n)) \geqslant 3$ by Lemmas 6, 7 of [10], (1) if $(u, v) \in H'$, then $\{f(u), f(v)\} \subset \{f(r), f(s)\}$, and (2) if $(t, u) \in H'$, $z_1 \in \overline{L(Q)}$, $z_2 \in (t, u)$, and $f(z_1) = f(z_2)$, then $f(z_2) \in \{f(r), f(s)\}$. Therefore $f(\bigcup H') \cup \{f(r), f(s)\}$ and $f(K - \bigcup H') \cup \{f(r), f(s)\}$ are two closed sets whose union is M and whose intersection is $\{f(r), f(s)\}$. Also, since $F(Q) \subset f(K - \bigcup H') \cup \{f(r), f(s)\}$ and M is a locally connected continuum, then $\mathrm{Bd}(C)$ is a subset of $F(Q)$ and therefore of $\{f(r), f(s)\}$.

LEMMA 4C. *If* $Q_1, Q_2, Q_3, \ldots$ *is a countable sequence of elements of* H *such that* $Q_1 \subset Q_2 \subset Q_3 \subset \ldots$, *then* $F(\bigcup Q_n) = \mathrm{Cl}(\bigcup F(Q_n))$.

Proof. Suppose that $A_1, A_2, \ldots$ is an ordering of the elements of $\bigcup Q_n$ and $A_1^p, A_2^p, \ldots$ is an ordering of the elements of Q_p ($p = 1, 2, \ldots$), as in

Lemma 4A. Also, as in Lemma 4A let $A_1, A_2, \ldots$ define $X_1, X_2, \ldots$ and let $A_1^p, A_2^p, \ldots$ define $X_1^p, X_2^p, \ldots$ $(p = 1, 2, \ldots)$.

Let p be a fixed positive integer. Since $A_3^p = A_q$ for some integer q, then $X_1^p \subset X_q$. Now suppose $X_j^p \subset X_r$. There is a positive integer s such that $A_{j+1}^p = A_s$. Therefore $X_{j+1}^p \subset X_{r+s}$, and $\bigcup X_j^p \subset \bigcup X_j$ by induction. Since p was arbitrary, we have $\bigcup_{p,n} X_n^p \subset \bigcup_n X_n$.

Now $A_3 = A_j^p$ for some p, j, so $X_1 \subset X_j^p$. Now suppose $X_r \subset X_n^k$ and let $A_j = A_{s_j}^{t_j}$ for $j = 1, \ldots, r+1$. Also let $t = \sup\{t_1, \ldots, t_{r+1}\}$. Since $Q_1 \subset Q_2 \subset Q_3 \subset \ldots$, then $A_i \in Q_t$ for $i = 1, \ldots, r+1$. Thus let $A_i = A_{a_i}^t$ for $i = 1, \ldots, r+1$. If $u = \sup\{a_1, \ldots, a_{r+1}\}$, then $X_{r+1} \subset X_u^t$. By induction $\bigcup_n X_n \subset \bigcup_{p,n} X_n^p$, and hence the two sets are equal.

Now

$$F(\bigcup_n Q_n) = f(\mathrm{Cl}(\bigcup_n X_n)) = f(\mathrm{Cl}(\bigcup_{p,n} X_n^p)).$$

But

$$\mathrm{Cl}(\bigcup_n F(Q_n)) = \mathrm{Cl}(\bigcup_n f(\mathrm{Cl}(\bigcup_p X_n^p))) = \mathrm{Cl}\Big(f\big(\bigcup_n(\mathrm{Cl}(\bigcup_p X_n^p))\big)\Big),$$

which is $f\big(\mathrm{Cl}(\bigcup_n(\mathrm{Cl}(\bigcup_p X_n^p)))\big)$ by the compactness of K and the continuity of f. Trivially

$$f\big(\mathrm{Cl}(\bigcup_n(\mathrm{Cl}(\bigcup_p X_n^p)))\big) = f(\mathrm{Cl}(\bigcup_{p,n} X_n^p)).$$

Definition. If $Q \in H$ let G_Q denote the set of all components g of $M - F(Q)$ such that $\bar{g} \cap P$ is void, and let G_Q' denote the set of all such g where $\bar{g} \cap P$ is not void.

LEMMA 4D. *Let $Q \in H$ and let $N = F(Q) \cup (\bigcup G_Q)$. Let L_Q (resp. L_Q') denote $\{x \mid x \in M - P$ and there is an element g_x of $G_Q(G_Q')$ such that $x \in \mathrm{Bd}(g_x)$. Then* (1) *if $x \in L_Q'$, then there are only finitely many elements g of G_Q' such that $x \in \mathrm{Bd}(g)$,* (2) *if y is a limit point of L_Q', then $y \in P$,* (3) *there is a separable open set U_Q such that $F(Q) - P \subset U_Q$, and $G_Q \cup G_Q'$ is countable,* (4) *N is a closed separable subset of M so that $N - P$ is locally connected, and* (5) *if C is a component of $M - N$ then* $\mathrm{card}(\mathrm{Bd}(C)) \leqslant 2$.

Proof. The proofs of (1), (2) are analogous, so we prove only (1). Suppose $\{g_1, g_2, \ldots\}$ is a countable infinite subset of G_Q' such that $x \in \mathrm{Bd}(g_i)$ for $i = 1, 2, \ldots$, and let $\mathrm{Bd}(g_i) = \{x, y_i\}$, $i = 1, 2, \ldots$ Since some subsequence of the y_i's converges to a point v, without loss of generality we suppose $y_1, y_2, \ldots$ converges to v. Let h: $M \to [0, 1]$ be a continuous map so that $h(x) = 0$ and $h(P) = 1$. There is a number t in $(0, 1)$ such that $h^{-1}(t) \cap \big(\{v\} \cup (\bigcup_1^\infty \mathrm{Bd}(g_i))\big)$ is void. Since each g_i is connected, there exists a point u_i in $g_i \cap h^{-1}(t)$, so let u be a limit point of $\{u_1, u_2, \ldots\}$.

Let U be a connected open set containing u such that

$$\bar{U} \subset M - (\{v\} \cup h^{-1}(\{0, 1\})),$$

and let n_0 be a positive integer so that if $i > n_0$ then $y_i \notin \bar{U}$. Let i, j be positive integers so that $i > j > n_0$ and $u_i, u_j \in U$. But $\mathrm{Bd}(g_i)$ separates g_i from g_j in M and U is a connected open set lying in $M - \mathrm{Bd}(g_i)$ and intersecting g_i and g_j. This involves a contradiction.

(3) Since $L(Q)$ is countable, $L(Q)$ intersects only countably many intervals I such that the endpoints of I lie in $f^{-1}(P)$, and $\mathrm{Int}(I) \cap f^{-1}(P)$ is void. Let $I_i = [a_i, b_i]$, $i = 1, 2, \ldots$, denote the set of all such intervals I. Also since $L(Q)$ is countable, there is a countable subset $y_1, y_2, \ldots$ of $K - f^{-1}(P)$ so that if $z \in \overline{L(Q)} \cap (a_k, b_k)$, then there exist y_i, y_j such that $a_k < y_i \leqslant z \leqslant y_j < b_k$. For each such pair i, j for which there exist such a z, k as above let $A(i, j)$ be a finite cover of $f([y_i, y_j])$ by connected open sets whose closures are compact metric and lie in $M - P$. Let $B(i, j) = \bigcup A(i, j)$ for each such pair i, j and let U_Q denote $\bigcup_{(i,j)} B(i, j)$. Clearly U_Q is a separable open set such that $F(Q) - P \subset U_Q$. Also since U_Q is separable and each element of $G_Q \cup G'_Q$ has a boundary point in $F(Q) - P$, then $G_Q \cup G'_Q$ is countable.

(4) Since each element g of G_Q has the property that $\bar{g} \subset M - P$, then $\bar{g}$ is covered by a finite number of metrizable open sets, and is thus separable and metrizable. Therefore N is separable and properties of local connectivity may be used to show that N is closed.

Now let $x \in N - P$ and U be an open set containing x. By (1), (2) there is an open set U' so that $U' \subset U - P$ and $U' \cap L'_Q \subset \{x\}$. Furthermore, if $x \in L'_Q$ then the elements g of G'_Q where $x \in \mathrm{Bd}(g)$ may be labeled $g_1, g_2, \ldots, g_n$. There is an open set U'' so that $x \in U'' \subset U'$ and $\mathrm{Bd}(g_i) \cap U'' = \{x\}$ for $i = 1, 2, \ldots, n$. Since M is locally connected at x, there is a connected open set V so that $x \in V \subset U''$. For each g_i $(1 \leqslant i \leqslant n)$, $V \cap g_i$ is a connected open set V_i so that $\{x\} \cup V_i$ is open in $\{x\} \cup g_i$. It is easily seen that $V - \bigcup_1^n V_i$ is a connected open subset of N contained in U. It follows in this case that N is locally connected at x. If $x \notin L'_Q$ then $x \in \mathrm{Int}(N)$, and the proof is trivial.

(5) The proof of (5) follows from Lemma 4B.

This completes the proof of Lemma 4D. We continue the proof of Theorem 4.

Let Q_1 be an element of H, let $M_1 = F(Q_1)$, and let $N_1 = M_1 \cup (\bigcup G_{Q_1})$. With the aid of Lemma 4D there is a separable open set U_{Q_1} such that (1) $N_1 - P \subset U_{Q_1}$, and (2) if $g \in G'_Q$ and there is an arc ab in $\bar{g} - P$ whose endpoints are the points a, b of $\mathrm{Bd}(g)$, then U_{Q_1} contains one such arc. Let Q_2 denote a countable set such that $Q_1 \subset Q_2$ and $F(Q_2) \supset \bar{U}_{Q_1}$. We define M_2 to be $F(Q_2)$ and $N_2 = M_2 \cup (\bigcup G_{Q_2})$.

Now suppose that countable well ordered sequences $Q_1, Q_2, \ldots; M_1,$

$M_2, \ldots; N_1, N_2, \ldots$ have been defined, but that $\bigcup N_i \not\supset M-P$. If there exists a last term Q_α, M_α, N_α, respectively, for each sequence define $Q_{\alpha+1}$, $M_{\alpha+1}$, $N_{\alpha+1}$ analogously to the way Q_2, M_2, N_2 were defined from Q_1, M_1, N_1.

Now suppose there is no last term, but that β is the first ordinal following 1, 2, ... Define $Q_\beta = \bigcup_{\alpha<\beta} Q_\alpha$, $M_\beta = F(Q_\beta)$ and $N_\beta = M_\beta \cup (\bigcup G_{Q_\beta})$.

If $\bigcup_{\alpha<\aleph_1} N_\alpha \not\supset M-P$ let $x \in M_1 - P$ and let $y \in M-(P \cup (\bigcup_{\alpha<\aleph_1} N_\alpha))$. Since $M-P$ is connected, then by use of a finite chain of connected metrizable open subsets of $M-P$, we find there is a metrizable arc xy from x to y in $M-P$. Since $xy \subset M-P$, then xy intersects $U_{Q_\alpha} - N_\alpha$. The set of all sets of the form $xy \cap (\mathrm{Int}(N_{\alpha+1}) - N_\alpha)$ would thus be an uncountable collection of mutually exclusive open subsets of a separable set, a contradiction.

Part (1) of the conclusions of Theorem 4 holds by construction and Lemma 4D. Part (2) holds by Lemma 4D. Parts (3) and (4) hold by the above remarks. We now show (5) holds.

(5i) If C is a component of $N_{a_i} - \mathrm{Cl}(\bigcup_{a<a_i} N_a)$ then by construction, Lemma 4C, and Lemma 4D, C has at most two boundary points $\{x, y\}$ in $\mathrm{Cl}(\bigcup_{a<a_i} N_a)$. By Lemma 4D C is locally connected at each point of $C-P$, so $\bar{C}$ is locally connected at each point except, for possibly points of $\{x, y\} \cup (\bar{C} \cap P)$, which is totally disconnected. By Theorem 2, if $\bar{C}$ fails to be locally connected at some point it fails to be locally connected et each point of some nondegenerate subcontinuum. Thus, $\bar{C}$ is locally connected.

Now N_{a_i} is separable, so let D be a countable subset of $f^{-1}(N_{a_i})$ so that $f(\bar{D}) = N_{a_i}$. By Lemma 2 of [9] $Z = \bar{D} \cap f^{-1}(C)$ is separable and so $f(Z) = C$ is separable. By Theorem 1 of [10], $\bar{C}$ is metrizable.

Suppose there is an open set U_3 containing R such that for each a_i there is an $a_j > a_i$ such that $\overline{C_{a_j}} \not\subset U_3$. Let U_2, U_1 be open sets such that $R \in U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U_3$. By applying the ideas of Theorem 2 we find a subsequence $b_1, b_2, \ldots$ of $a_1, a_2, \ldots$ and (1) a sequence $u_{b_1}, u_{b_2}, \ldots$ converging to a point u of $\mathrm{Bd}(U_1)$, (2) a sequence $v_{b_1}, v_{b_2}, \ldots$ converging to a point v of $\mathrm{Bd}(U_2)$ and (3) a sequence $w_{b_1}, w_{b_2}, \ldots$ converging to a point w of $\mathrm{Bd}(U_3)$, where (4) $u_{b_i}, v_{b_i}, w_{b_i} \in C_{b_i}$ for each i. Let U, V, W be disjoint connected open sets containing u, v, w, respectively. For some j C_{b_j}, $C_{b_{j+1}}$ each intersect all three of U, V, W. The component S of $M - N_{b_j}$ which contains $C_{b_{j+1}}$ has at most two boundary points and one of U, V, W (say V) contains none of these. But $\overline{S \cap V} \cap \overline{V-S}$ does contain a boundary point of S, a contradiction.

Now let $R \in U = U^0$ and $R \notin P$. Also let $R \in V = V^0 \subset \bar{V} \subset U-P$ and let $\overline{C_{a_j}} \subset V$ for $i \leqslant j$. Since $\overline{C_{a_j}}, \overline{C_{a_k}}$, P are disjoint for $j > i$ and $|j-k| \geqslant 2$, then using Theorem 4 (5i) it follows that $C_{a_i} \cup (\bigcup_{j>i} \overline{C_{a_j}}) \cup \{R\}$ is both open in and has at most a two point boundary relative to $\{R\} \cup (\bigcup_{a_j \in B} \overline{C_{a_j}})$.

This completes the proof of Theorem 4.

THEOREM 5. *If P is totally disconnected, then M is the continuous image of an arc.*

Proof. We first assume that $M-P$ is connected and let Q_α, M_α, N_α, $\alpha \in A$, be as in Theorem 4. We recall that if $\alpha \in A$ and C is a component of $M-N_\alpha$ such that (1) $\bar{C} \cap P$ is not void, (2) C has boundary points a, b in N_α such that there is an arc ab in $\bar{C}-P$ from a to b, then $Q_{\alpha+1}$ is chosen so that its closure contains such an arc.

Since N_1 is separable and locally connected at each point of N_1-P, there are only countably many components of N_1 which contain a point of N_1-P, so let these components be labeled C_{11}, $C_{12}, \ldots$ We note also that (1) Theorem 4 implies that each C_{1j} is a locally connected separable metric continuum, and (2) the local connectivity of N_1-P implies that if U is an open set containing P, then U contains all but finitely many C_{1j}'s. Lemma 4D helps imply that $P \cup L'_{Q_1}$ is a totally disconnected closed set, so by Theorem 3 there is a tree T_{1j} lying in C_{1j} and containing $(P \cup L'_{Q_1}) \cap C_{1j}$ for $j = 1, 2, \ldots$ Let $T_1 = P \cup (\bigcup_j T_{1j})$ and note that (2) above implies that T_1 is closed.

Let C_{21}, $C_{22}, \ldots$ denote the set of all closures of components of N_2-N_1 which contain a point of $M-P$. As above (1) each C_{2j} is a locally connected separable metric continuum and (2) if U is an open set containing P, then U contains all but finitely many C_{2j}'s. For each j let T_{2j} denote a tree in C_{2j} containing $C_{2j} \cap (P \cup L'_{Q_1} \cup L'_{Q_2})$ and let T_2 denote the closed set $T_1 \cup (\bigcup_j T_{2j})$.

Suppose C_{ij}, T_{ij}, T_i have been defined for i in an initial segment s of A, and β is the first term of A following each term of s.

Case 1. Suppose β has an immediate predecessor $\beta-1$. Analogously to the case $\beta = 2$ let $C_{\beta 1}$, $C_{\beta 2}, \ldots$ be the closures of components of $N_\beta - N_{\beta-1}$ which contain a point of $M-P$. For each j let $T_{\beta j}$ be a tree in $C_{\beta j}$ containing $C_{\beta j} \cap (P \cup L'_{Q_{\beta-1}} \cup L'_{Q_\beta})$, and let T_β denote the closed set $T_{\beta-1} \cup (\bigcup_j T_{\beta j})$.

Case 2. Suppose β has no immediate predecessor. Recall that $Q_\beta = \bigcup_{\alpha<\beta} Q_\alpha$. We let $C_{\beta 1}$, $C_{\beta 2}, \ldots$ be the closures of the elements of G_{Q_β}. For each j let $T_{\beta j}$ be a tree in $C_{\beta j}$ containing $\mathrm{Bd}(C_{\beta j})$. Let $T_\beta = (\bigcup_{\alpha<\beta} T_\alpha) \cup (\bigcup_j T_{\beta j}) \cup \mathrm{Bd}(M_\beta)$.

We note that $\mathrm{Bd}(M_\beta)$ is totally disconnected, for if not, there is a nondegenerate continuum X lying in $(M-P) \cap \mathrm{Bd}(M_\beta)$. Let x_1, x_2, x_3 be three points of X and let U_1, U_2, U_3 be disjoint connected open sets lying in $M-P$ such that $x_p \in U_p$, $p = 1, 2, 3$. Let y_i $(i = 1, 2, 3)$ denote a point of $\mathrm{Int}(M_\beta) \cap U_i$ and let $x_i y_i$ denote an arc from x_i to y_i in U_i. Let m denote the least index t such that each of y_1, y_2, y_3 belongs to N_t, and let g_i denote the component of $x_i y_i - N_m$ containing x_i. But $X \cup (\bigcup_1^3 g_i)$ lies in a component of $M - N_m$ having at least three boundary points in N_m, a contradiction.

We also note that the local connectivity of $N_\beta - P$ implies that $P \cup \mathrm{Bd}(M_\beta) \cup (\bigcup_j T_{\beta j})$ is closed. If x is a limit point of T_β, then $x \in \mathrm{Int}(M_\beta)$ implies x is a limit point of T_α for some $\alpha < \beta$, so $x \in T_\alpha \subset T_\beta$. If $x \in M - \mathrm{Int}(M_\beta)$ then x is an element of $P \cup \mathrm{Bd}(M_\beta) \cup (\bigcup_j T_{\beta j})$. Therefore T_β is closed.

We define $T = \bigcup_\alpha T_\alpha$ and proceed to show that T is closed, connected, and locally peripherally finite.

First let U be an open set containing P. Since each point of $M - U$ belongs to some $\mathrm{Int}(N_a - P)$, then $M - U$ is covered by a finite number of such sets. Since $\{N_a, a \in A\}$ is monotone, $M - U \subset \mathrm{Int}(N_b - P)$ for some index b. Since $T \cap (M - U) = T_b \cap (M - U)$ it suffices to note that each such T_b is closed, for if $x \in \bar{T} - T$, then U could be chosen so that $x \in M - U$.

To show that T is connected let $x \in T_{11} - P$ and suppose y belongs to some $T_{b_j} - P$. There is an arc xy from x to y lying in $M - P$. Let W denote the set of components of $xy - ((\bigcup L'_{Q_\alpha}) \cup Y)$, where $Y = \bigcup \{\mathrm{Bd}(M_\beta) \mid \beta$ is a limit ordinal in $A\}$. Each w in W is a subset of a set $C_{i_w k_w}$, where $T_{i_w k_w}$ contains the endpoints of w. Thus $(xy - \bigcup W) \cup (\bigcup_{w \in W} T_{i_w k_w})$ is a connected subset of T containing x, y. Since the component of T containing x contains $T - P$ and since $P \subset \overline{T - P}$, then T is connected.

We now show T is locally peripherally finite. Now $J = Y \cup P \cup (\bigcup L'_{Q_\alpha})$ is a totally disconnected closed subset of T. If $x \in T - J$ then $x \in \mathrm{Int}(C_{ij}) \cap T$ for some i, j. But $T \cap \mathrm{Int}(C_{ij}) = T_{ij} \cap \mathrm{Int}(C_{ij})$ and T_{ij} is locally peripherally finite at x, and T is also. Let $x \in J$ and let U be an open set containing x. Since J is totally disconnected, there is an open set V so that $x \in V \subset \bar{V} \subset U$ and $J \cap \mathrm{Bd}(V)$ is void. There is a collection $V_1, \ldots, V_n$ of open sets so that (1) $V_1, \ldots, V_n$ covers $\mathrm{Bd}(V)$, (2) each $T \cap \mathrm{Bd}(V_i)$ is finite, $1 \leqslant i \leqslant n$, and (3) each $V_i \subset U - J$. If $R' = V \cup (\bigcup_1 V_i)$, then $x \in R' \subset U$ and $T \cap \mathrm{Bd}(R')$ is finite.

By Theorem 7 of [12], T is the image of an arc I under a continuous map g.

For each set C_{ij} as defined above we let x_{ij} denote a point of I such that $g(x_{ij}) \in C_{ij}$. For each such pair i, j let $I_{ij} = [0, 1] \times \{(i, j)\}$ and let g_{ij}: $I_{ij} \to C_{ij}$ be a continuous onto map such that $g_{ij}(p \times (i, j)) = g(x_{ij})$, $p = 0, 1$. For each i, j replace x_{ij} by I_{ij} in order to form a space J' with the obvious order and the interval topology. Define a map $\bar{g}$: $J' \to M$ so that $\bar{g}(x) = g(x)$ if $x \notin I_{ij}$ for any i, j and $\bar{g}(x) = g_{ij}(x)$ if $x \in I_{ij}$.

Since $M = T \cup (\bigcup_{i,j} C_{ij})$, then $\bar{g}$ is clearly onto. We need only show that $\bar{g}$

is continuous. Since $\bar{g}$ is continuous on $\mathrm{Cl}(J'-(\bigcup_{i,j} I_{ij}))$ if $\bar{g}$ is not continuous, there are a point x of J' and an open set U containing $\bar{g}(x)$ so that if V is an open set containing x there is a set I_{ij} so that $I_{ij} \subset V$ and $\bar{g}(I_{ij}) \not\subset U$. Let R_1 denote an open set so that $\bar{g}(x) \in R_1 \subset \bar{R}_1 \subset U$, and for each i, j let $\{a_{ij}, b_{ij}\}$ denote $C_{ij} \cap N_{i-1}$ if i has an immediate predecessor and denote $M_i \cap C_{ij}$, otherwise. By using f^{-1} and the fact that every infinite sequence in J' has a countable convergent subsequence, we may assume without loss of generality that there is an infinite sequence $(i_1, j_1), (i_2, j_2), \ldots$ such that (1) $\{a_{i_n j_n}\}_{n=1}^{\infty}$ converges to a point a', (2) $\{b_{i_n j_n}\}_{n=1}^{\infty}$ converges to a point b', (3) each $C_{i_n j_n}$ intersects R_1 and $M-U$ and (4) $\{i_n\}_{n=1}^{\infty}$ is either constant or increasing (since only a finite number of steps may be taken backwards in a well ordered sequence).

Case 1. Assume $i_1 < i_2 < i_3 < \ldots$ We apply Theorem 4 (5) to find that there is a positive integer N so that if $n > N$ then $C_{i_n j_n} \subset W$, where W is an open set lying in U if $a' \in \bar{R}_1$, and W is an open set in $M-\bar{R}_1$ if $a' \notin \bar{R}_1$. This involves a contradiction.

Case 2. Assume $i_1 = i_2 = i_3 = \ldots$ and i_1 is not a limit ordinal. Since each open set containing P contains all but finitely many $C_{i_n j_n}$'s, there is a point z of the limiting set of the $C_{i_n j_n}$'s in P. Since $z \in P$ let W, X be disjoint open sets covering $P \cup \{a', b'\}$ so that $z \in W$ and W is as in Case 1. Since an infinite number of the $C_{i_n j_n}$'s intersect W and lie in $W \cup X$, they lie in W, a contradiction.

Suppose i_1 is a limit ordinal. The limiting set of the $C_{i_n j_n}$'s is a nondegenerate continuum X, so let $x \in X-(P \cup \mathrm{Bd}(M_{i_1}))$. Since N_{i_1} is locally connected at x there is a connected open set W of N_{i_1} which contains x and no point of $P \cup \mathrm{Bd}(M_{i_1})$. Thus, W is a subset of a single C_{ij} and intersects infinitely many of them, a contradiction. Therefore $\bar{g}$ is continuous and the case where $M-P$ is connected is now complete.

Suppose $M-P$ is not connected and $\{C_\alpha : \alpha \in A\}$ is the set of components of $M-P$. For each $\alpha \in A$ let T_α be a nondegenerate locally peripherally finite continuum in $\bar{C}_\alpha$ containing $P \cap \bar{C}_\alpha$. As in the arguments above the local connectivity of M may be used to show $T' = \bigcup_{\alpha \in A} T_\alpha$ is a continuum which is locally peripherally finite at each point of $T'-P$, and thus, by an argument analogous to the one that T is locally peripherally finite at each point P, so is T'. By Ward [12] T' is the continuous image of an arc A under a map g. For each $\alpha \in A$ let $y_\alpha \in T_\alpha - P$ and let $x_\alpha \in g^{-1}(y_\alpha)$.

Now with the aid of the first part we find for each $\alpha \in A$ an arc B_α and a continuous onto map g_α: $B_\alpha \to \bar{C}_\alpha$, where (1) if $B_\alpha = [a_\alpha, b_\alpha]$ then $g_\alpha(a_\alpha) = g_\alpha(b_\alpha) = y_\alpha$, and (2) if $\alpha \neq \alpha'$ then $B_\alpha \cap B_{\alpha'}$ is void. We now construct an arc J by replacing each x_α in B by B_α and giving J the interval topology. We define a map $\bar{g}$: $J \to M$ by $\bar{g}(x) = g(x)$ if $x \in B - \{x_\alpha, \alpha \in A\}$ and $\bar{g}(x) = g_\alpha(x)$

if $x \in B_x$. $\bar{g}$ is clearly onto, and the continuity of $\bar{g}$ is established much the same way as in the case where $M-P$ is connected. This completes the proof of Theorem 5.

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