COLLOQUIUM MATHEMATICUM

VOL. LVI 1988 FASC. 2

ON SOME GENERIC MINIMAL SUBMANIFOLDS OF AN ODD DIMENSIONAL SPHERE

BY

MASAHIRO KON (HIROSAKI)

1. Introduction. Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere. We know that S^{2m+1} admits a standard Sasakian structure (cf. [5]). We denote by (ϕ, ξ, η, g) the structure tensors of S^{2m+1}. Let M be an $(n+1)$-dimensional submanifold isometrically immersed in S^{2m+1}. We denote by the same g the Riemannian metric tensor field induced on M from that of S^{2m+1}. Throughout this paper, we assume that the submanifold M is tangent to the structure vector field ξ of S^{2m+1}. When the transform of the normal space $T_x(M)^{\perp}$ of M by ϕ is always tangent to M, that is,

$$\phi T_x(M)^{\perp} \subset T_x(M) \quad \text{for any } x \in M,$$

$T_x(M)$ being the tangent space of M, then M is called a *generic* submanifold of S^{2m+1} (cf. [4]).

For any vector field X tangent to M, we put $\phi X = PX + FX$, where PX is the tangential part of ϕX, and FX the normal part of ϕX. Then P is an endomorphism on the tangent bundle $T(M)$. We denote by S the Ricci tensor of a generic submanifold M of S^{2m+1}. If S satisfies

$$S(\phi^2 X, \phi^2 Y) = ag(\phi^2 X, \phi^2 Y) + bg(P\phi^2 X, P\phi^2 Y),$$

where a and b are constant, then M is called a *pseudo-Einstein* generic submanifold (cf. [3]). We notice that the subspace of $T_x(M)$ orthogonal to ξ is spanned by all vectors $\phi^2 X$, where $X \in T_x(M)$. Let $S^m(r)$ be an m-dimensional sphere with radius r and RP^m be the real projective space of real dimension m. We denote by (S^1, RP^m) the circle bundle over RP^m.

The purpose of the present paper is to prove the following

THEOREM. *Let M be a compact orientable $(n+1)$-dimensional generic minimal submanifold of S^{2m+1}. If the Ricci tensor S of M satisfies*

$$S(\phi^2 X, \phi^2 X) \geqslant (n-1) g(\phi^2 X, \phi^2 X),$$

then M is (S^1, RP^m) $(n = m)$ or M is the pseudo-Einstein hypersurface $S^m(r) \times S^m(r)$ $(r = \sqrt{1/2},\ n = 2m)$ of S^{2m+1}.

2. Preliminaries. Let M be an $(n+1)$-dimensional submanifold of S^{2m+1}. The operator of covariant differentiation with respect to the Levi–Cività connection in S^{2m+1} (resp. M) will be denoted by $\bar{\nabla}$ (resp. ∇). Then the Gauss and Weingarten formulas are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \quad \text{and} \quad \bar{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle of M from that of S^{2m+1}. A and B appearing here are both called the *second fundamental form* of M and are related by

$$g(B(X, Y), V) = g(A_V X, Y).$$

For any normal vector field V, A_V is a symmetric linear transformation on $T_x(M)$. If $\operatorname{Tr} A_V = 0$ for any normal vector V, then M is said to be *minimal*. We see that M is minimal if and only if $\operatorname{Tr} B = 0$. Let R denote the Riemannian curvature tensor of M. Then the Gauss equation is given by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_a g(A_a Y, Z)A_a X - \sum_a g(A_a X, Z)A_a Y,$$

where we have put $A_a = A_{v_a}$, $\{v_a\}$ being an orthonormal frame for $T_x(M)^\perp$. We define the covariant derivative $\nabla_X A$ of the second fundamental form A by

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A\nabla_X Y.$$

If $\nabla_X A = 0$ for all X, then the second fundamental form A is said to be *parallel*. The Codazzi equation of M is given by

$$(\nabla_X A)_V Y = (\nabla_Y A)_V X.$$

Since the structure vector field ξ is tangent to M, for any vector field X tangent to M we have

$$\bar{\nabla}_X \xi = \phi X = \nabla_X \xi + B(X, \xi),$$

whence

$$PX = \nabla_X \xi \quad \text{and} \quad FX = B(X, \xi).$$

Moreover, we obtain

$$A_V \xi = -\phi V \quad \text{and} \quad \nabla_X \phi V = -PA_V X + \phi D_X V.$$

We notice here that the second fundamental form A of M satisfies

$$A_{FX}Y = A_{FY}X$$

for any vector fields X, Y in $\phi T(M)^{\perp}$ (see [3], p. 169).

If the second fundamental form B of M is of the form

$$B(X, Y) = \eta(X)FY+\eta(Y)FX$$

for any vector fields X and Y tangent to M, then M is said to be *contact totally geodesic*. We easily see that M is contact totally geodesic if and only if $B(\phi^2 X, \phi^2 Y) = 0$ for all X and Y. We see that (S^1, RP^m) is contact totally geodesic in S^{2m+1}. The second fundamental form of (S^1, RP^m) is parallel. The Ricci tensor S of (S^1, RP^m) is given by

$$S(\phi^2 X, \phi^2 Y) = (n-1)g(\phi^2 X, \phi^2 Y).$$

Thus (S^1, RP^m) is a pseudo-Einstein anti-invariant submanifold of S^{2m+1} (see [5], p. 343).

If M is a compact hypersurface with parallel second fundamental form of S^{2m+1}, then M is $S^p(r_1)\times S^q(r_2)$ (cf. [4], p. 40). Furthermore, if M is a pseudo-Einstein minimal hypersurface satisfying the condition on the Ricci tensor S in the Theorem, then M is congruent to $S^m(r)\times S^m(r)$ $(r = \sqrt{1/2})$ (see [3] and [4]).

We use the following lemma:

LEMMA ([2]). *Let M be an $(n+1)$-dimensional minimal submanifold of S^{2m+1}. Then*

$$\tfrac{1}{2}\Delta\Big(\sum_a \operatorname{Tr} A_a^2\Big) = (n+1)\sum_a \operatorname{Tr} A_a^2 - \sum_{a,b}(\operatorname{Tr} A_a A_b)^2 + \sum_{a,b}\operatorname{Tr}[A_a, A_b]^2 + g(\nabla A, \nabla A).$$

Since the $(2m-n, 2m-n)$-matrix $(\operatorname{Tr} A_a A_b)$ is symmetric, it can be diagonalized for a suitable choice of a frame $\{v_a\}$ at each point of M so that

$$\sum(\operatorname{Tr} A_a A_b)^2 = \sum(\operatorname{Tr} A_a^2)^2.$$

3. Proof of the Theorem. From the equation of Gauss, the Ricci tensor S of M is given by

$$S(X, Y) = ng(X, Y) - \sum_a g(A_a^2 X, Y).$$

In accordance with the assumption on the Ricci tensor S, we see that

$$g(\phi^2 X, \phi^2 X) \geqslant \sum_a g(A_a\phi^2 X, A_a\phi^2 X) \geqslant 0, \tag{3.1}$$

whence

$$g(V, V) \geqslant \sum_a g(A_a\phi V, A_a\phi V) \geqslant 0.$$

We now take an orthonormal frame $\{e_1, \dots, e_n, \xi\}$ of $T_x(M)$. Then we find

$$\begin{aligned}\sum_a g(A_a\phi V, A_a\phi V) &= \sum_{a,i} g(A_a\phi V, e_i)g(A_a\phi V, e_i) + \sum_a g(A_a\phi V, \xi)g(A_a\phi V, \xi)\\ &= \sum_{a,i} g(A_a\phi V, e_i)g(A_a\phi V, e_i) + g(V, V).\end{aligned}$$

Therefore we obtain

$$\sum_{a,i} g(A_a\phi V, e_i)g(A_a\phi V, e_i) = 0,$$

and hence $g(A_a\phi V, e_i) = 0$ for all a and i, which means that

$$g(A_a\phi V, \phi^2 X) = 0$$

for any vector field X tangent to M or, equivalently,

$$g(A_a\phi V, X) - \eta(X)g(A_a\phi V, \xi) = 0.$$

Thus we have

$$A_a\phi V = g(A_a\phi V, \xi)\xi = -g(\phi V, \phi v_a)\xi = -g(V, v_a)\xi. \tag{3.2}$$

On the other hand, (3.1) implies

$$\begin{aligned} n = \sum_i g(e_i, e_i) &\geqslant \sum_{a,i} g(A_a e_i, A_a e_i)\\ &= \sum_{a,i,j} g(A_a e_i, e_j)g(A_a e_i, e_j) + \sum_{a,i} g(A_a e_i, \xi)g(A_a e_i, \xi)\\ &= \sum_{a,i,j} g(A_a e_i, e_j)g(A_a e_i, e_j) + p,\end{aligned}$$

where p denotes the codimension of M, that is, $p = 2m-n$, which reduces to

$$n-p \geqslant \sum_{a,i,j} g(A_a e_i, e_j)g(A_a e_i, e_j). \tag{3.3}$$

We can take an orthonormal frame $\{e_1, \dots, e_{n-p}, \phi v_1, \dots, \phi v_p, \xi\}$ of $T_x(M)$ such that $\{v_1, \dots, v_p\}$ forms an orthonormal frame of $T_x(M)^{\perp}$. Then, by (3.2), A_a is represented by a matrix form

$$A_a = \begin{bmatrix} h^a_{ts} & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}, \tag{3.4}$$

where $h^a_{ts} = g(A_a e_t, e_s)$ $(t, s = 1, \dots, n-p)$, $g(A_a\phi v_b, \xi) = -\delta_{ab}$, δ_{ab} being the

Kronecker delta. For each a we put

$$H_a = \left[\begin{array}{c|c} h^a_{ts} & 0 \\ \hline 0 & 0 \end{array}\right],$$

which is a symmetric (n, n)-matrix. By a straightforward computation we obtain

$$\operatorname{Tr} A_a^2 = \operatorname{Tr} H_a^2 + 2, \qquad \operatorname{Tr} A_a^4 = \operatorname{Tr} H_a^4 + 2,$$

$$\sum_{a,b} \operatorname{Tr}[A_a, A_b]^2 = \sum_{a,b} \operatorname{Tr}[H_a, H_b]^2 - 2p(p-1),$$

$$(\operatorname{Tr} A_a^2)^2 = (\operatorname{Tr} H_a^2)^2 + 4\operatorname{Tr} H_a^2 + 4.$$

Substituting these equations into the formula of the Lemma, we find

$$(3.5) \quad g(\nabla A, \nabla A) - \tfrac{1}{2}\Delta\Big(\sum_a \operatorname{Tr} A_a^2\Big) = \sum_a (\operatorname{Tr} H_a^2)^2 + 4\sum_a \operatorname{Tr} H_a^2 + 4p$$

$$- \sum_{a,b} \operatorname{Tr}[H_a, H_b]^2 + 2p(p-1) - (n+1)\sum_a \operatorname{Tr} H_a^2 - 2(n+1)p.$$

On the other hand, we can choose $\{e_1, \ldots, e_{n-p}\}$ such that $h^a_{ts} = 0$ when $t \neq s$. We put $h^a_t = h^a_{tt}$ $(t = 1, \ldots, n-p)$. Then

$$-\sum_b \operatorname{Tr}[H_a, H_b]^2 = \sum_{b \neq a, t, s} (h^b_{ts})^2 (h^a_t - h^a_s)^2 \leqslant 4 \sum_{b \neq a, t, s} (h^b_{ts})^2 (h^a_s)^2.$$

From (3.1) we have

$$\sum_{b \neq a, t} (h^b_{ts})^2 \leqslant g(\phi^2 e_s, \phi^2 e_s) - (h^a_s)^2 = 1 - (h^a_s)^2$$

for each s. Thus we have

$$-\sum_b \operatorname{Tr}[H_a, H_b]^2 \leqslant 4\operatorname{Tr} H_a^2 - 4\operatorname{Tr} H_a^4.$$

Since $\operatorname{rank} H_a \leqslant n-p$ for each a, we obtain

$$(\operatorname{Tr} H_a^2)^2 \leqslant (n-p)\operatorname{Tr} H_a^4.$$

If $n = p$, that is, if M is anti-invariant in S^{2m+1}, then $H_a = 0$ for all a. Therefore, M is a contact totally geodesic anti-invariant submanifold of S^{2m+1}. Since the second fundamental form of M is parallel and is given by $H_a = 0$ in (3.4), we see that M is (S^1, RP^m) by the fundamental theorem of submanifolds (cf. [1], p. 207).

Next we assume that $n > p$. Then

$$-\sum_b \operatorname{Tr}[H_a, H_b] \leqslant 4\operatorname{Tr} H_a^2 - \frac{4}{n-p}(\operatorname{Tr} H_a^2)^2$$

for each a. Suppose now that $n-p-4 \geqslant 0$. Then (3.5) and the inequality above imply

$$g(\nabla A,\ \nabla A)-\tfrac{1}{2}\Delta\big(\sum_a \operatorname{Tr} A_a^2\big) \leqslant \frac{n-p-4}{n-p}\big[\sum_a (\operatorname{Tr} H_a^2)^2-(n-p)\sum_a \operatorname{Tr} H_a^2\big]$$
$$-(p-3)\sum_a \operatorname{Tr} H_a^2-2p(n-p)$$
$$\leqslant \frac{n-p-4}{n-p}\big[\sum_a \operatorname{Tr} H_a^2-(n-p)\big]\big(\sum_a \operatorname{Tr} H_a^2\big)$$
$$-(p-3)\sum_a \operatorname{Tr} H_a^2-2p(n-p),$$

whence

$$\int_M \big[g(\nabla A,\ \nabla A)+(p-3)\sum_a \operatorname{Tr} H_a^2+2p(n-p)\big]^* 1$$
$$\leqslant \frac{n-p-4}{n-p}\int_M \big[\sum_a \operatorname{Tr} H_a^2-(n-p)\big]\big(\sum_a \operatorname{Tr} H_a^2\big)^* 1.$$

From (3.3) we see that

$$n-p \geqslant \sum_a \operatorname{Tr} H_a^2.$$

Thus the right-hand side of the inequality above is non-positive.

If $p \geqslant 3$, then $\nabla A = 0$ and $n = p$. Since $p < n$, this is a contradiction. If $p = 2$, then

$$(p-3)\sum_a \operatorname{Tr} H_a^2+2p(n-p) \geqslant -(n-2)+4(n-2) = 3(n-2) > 0.$$

Again this is a contradiction. If $p = 1$, then

$$(p-3)\sum_a \operatorname{Tr} H_a^2+2p(n-p) \geqslant -2(n-1)+2(n-1) = 0.$$

Consequently, we obtain $\nabla A = 0$, that is, the second fundamental form of M is parallel. Moreover, we see that

$$\sum_a \operatorname{Tr} A_a^2 = n+1 = 2m$$

and M is a pseudo-Einstein hypersurface of S^{2m+1}. Then M is $S^m(r)\times S^m(r)$ $(r = \sqrt{1/2})$.

We next suppose that $n-p-4 < 0$. Then $n-4 < p < n$. Since $n+p$ is even, it follows that $p = n-2$. On the other hand, (3.2) implies

$$(\nabla_X A)_a \phi v_a + A_{D_X v_a}\phi v_a - A_a P A_a X + A_a \phi D_X v_a$$
$$= -g(D_X v_a,\ v_a)\xi - g(v_a,\ D_X v_a)\xi - g(v_a,\ v_a)\nabla_X \xi.$$

From this, using (3.2), we obtain

$$g\big((\nabla_X A)_a Y, \phi v_a\big) = g\big((\nabla_X A)_a \phi v_a, Y\big) = g(A_a PA_a X, Y) - g(PX, Y).$$

Therefore, the Codazzi equation implies that

$$g(A_a PA_a X, Y) - g(PX, Y) = 0.$$

Putting $Y = PX$, we get

$$g(A_a PA_a X, PX) = g(PX, PX).$$

Since $p = n-2$, the holomorphic subspace $PT_x(M)$ is spanned by X and PX, where X is a unit vector in $PT_x(M)$ such that $A_a X = \lambda X$. We can take such a vector X by (3.4). From the minimality of M we obtain $A_a PX = -\lambda PX$. We then have

$$g(A_a PA_a X, PX) = -\lambda^2 g(PX, PX) = g(PX, PX),$$

which implies $\lambda^2 = 0$, and hence $PX = 0$. This is a contradiction. From these considerations we have our assertion.

REFERENCES

[1] R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York 1964.
[2] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. 88 (1968), pp. 62–105.
[3] K. Yano and M. Kon, *Generic submanifolds of Sasakian manifolds*, Kodai Math. J. 3 (1980), pp. 163–196.
[4] – *CR Submanifolds of Kaehlerian and Sasakian Manifolds*, Birkhäuser Verlag, Boston 1983.
[5] – *Structures on Manifolds*, World Scientific, Singapore 1984.

HIROSAKI UNIVERSITY
HIROSAKI, 036
JAPAN

Reçu par la Rédaction le 20.3.1987